# THE NEW HAVEN MATHEMATICAL COLLOQUIUM

LECTURES DELIVERED BEFORE MEMBERS OF THE AMERICAN MATHEMATICAL SOCIETY IN CONNECTION WITH THE SUMMER MEETING, HELD SEPTEMBER 5TH TO 8TH, 1906, UNDER THE AUSPICES OF YALE UNIVERSITY

BY

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# PREFACE.

The American Mathematical Society held its Fifth Colloquium in connection with its Thirteenth Summer Meeting, under the auspices of Yale University, during the week September 3–8, 1906. At this Colloquium the following courses of lectures were given:

Professor Eliakim H. Moore, of the University of Chicago: three lectures on "The Theory of Bilinear Functional Operators."

Professor Ernest J. Wilczynski, of the University of California: three lectures on "Projective Differential Geometry."

Professor Max Mason, of Yale University: three lectures on "Selected Topics in the Theory of Boundary Value Problems of Differential Equations."

The present volume contains these lectures as prepared by their respective authors for publication.

The aim and scope of the Colloquia of the American Mathematical Society are well set forth in the following extract taken from an official circular issued prior to the first Colloquium:*

"The objects now attained by the Summer Meeting are twofold: an opportunity is offered for presenting before discriminating and interested auditors the results of research in special fields, and personal acquaintance and mutual helpfulness are promoted among the members in attendance. These two are the prime objects of such a gathering. It is believed however that a third no less desirable result lies within reach. From the concise, unrelated papers presented at any meeting only few derive substantial benefit. The mind of the hearer is too unprepared, the impression is of too short duration to produce accurate nowledge of either the content or the method. . . . Positive and exact knowledge, scientific knowledge, is rarely increased in these short and stimulating sessions.

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*Cf. Bull. Amer. Math. Soc., ser. 2, vol. 3 (1896), p. 49.

"On the other hand, the courses of lectures in our best universities, even with topics changing at intervals of a few weeks, do give exact knowledge and furnish a substantial basis for reading and investigation. . . .

"To extend the time of a lecture to two hours, and to multiply this time by three or by six, would be practicable within the limits of one week. An expert lecturer could present, in six two-hour lectures, a moderately extensive chapter in some one branch of mathematics. With some new matter, much that is old could be mingled, including for example digests of recent or too much neglected publications. There would be time for some elementary details as well as for more profound discussions. In short, lectures could be made profitable to all who have a general knowledge of the higher mathematics."

The colloquia preceding the fifth are the following :

## I. The Buffalo Colloquium, 1896.

(*a*) Professor Maxime Bôcher, of Harvard University : *Linear Differential Equations, and their Applications.*

This colloquium has not been published, but several papers appeared at about the time of the colloquium, in which the author dealt with topics treated in the lectures.*

(*b*) Professor James Pierpont, of Yale University : *Galois's Theory of Equations.*

This colloquium was published in the Annals of Mathematics, ser. 2, vols. 1 and 2 (1900).

## II. The Cambridge Colloquium, 1898.

(*a*) Professor William F. Osgood, of Harvard University : *Selected Topics in the Theory of Functions.*

This colloquium was published in the Bulletin of the Amer. Math. Soc., ser. 2, vol. 5 (1898), p. 59.

(*b*) Professor Arthur G. Webster, of Clark University : *The Partial Differential Equations of Wave Propagation.*

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## III. THE ITHACA COLLOQUIUM, 1901.

(*a*) Professor OSKAR BOLZA, of the University of Chicago: *The Simplest Type of Problems in the Calculus of Variations.*

Published in amplified form under the title: *Lectures on the Calculus of Variations,* Chicago, 1904.

(*b*) Professor ERNEST W. BROWN, of Haverford College: *Modern Methods of Treating Dynamical Problems, and in Particular the Problem of Three Bodies.*

## IV. THE BOSTON COLLOQUIUM, 1903.

(*a*) Professor HENRY S. WHITE, of Northwestern University: three lectures on *Linear Systems of Curves on Algebraic Surfaces.*

(*b*) Professor FREDERICK S. WOODS, of the Massachusetts Institute of Technology: three lectures on *Forms of Non-Euclidean Space.*

(*c*) Professor EDWARD B. VAN VLECK, of Wesleyan University; six lectures on *Selected Topics in the Theory of Divergent Series and Continued Fractions.*

This colloquium was published in full, under the title: *The Boston Colloquium. Lectures on Mathematics.* The Macmillan Co., N. Y., 1905.

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# INTRODUCTION TO A FORM OF GENERAL ANALYSIS.

BY

ELIAKIM HASTINGS MOORE.

## *Preface.*

Especially during the last decade the study of Integral Equations has brought to light numerous *analogies* between the $n$-fold algebra of real $n$-dimensional space and the theory of continuous functions of an argument varying over a finite interval of the real number system and the theory of certain types of functions of infinitely many variables. These analogies have their root in the classic analogy of a definite integral to an algebraic sum.

We lay down a fundamental principle of generalization by abstraction:

> *The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.*

This formulation was the dominant note of my series of lectures: *On the Theory of Bilinear Functional Operations,* at the Colloquium of the American Mathematical Society, held in September, 1906, in New Haven under the auspices of Yale University. In the note: *On the theory of systems of integral equations of the second kind,** read December 29, 1905, before the Chicago Section of the American Mathematical Society, I gave a preliminary indication of the point of view of the Colloquium Lectures.

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*In abstract, Bulletin of the American Mathematical Society, ser. 2, vol. 12 (1906), pp. 280, 283-4.—A brief synopsis of the Colloquium Lectures will shortly be published in the Bulletin.

The subsequent development of that point of view led to the determination of a form of General Analysis * to which the present memoir is an introduction. Intending in this preface to give a summary characterization and a precise definition of this form of General Analysis, for the purpose of orientation we first call to mind a few of the types of *variables* $p$, *of ranges* (classes, sets, *ensembles*, *Mengen*) $\mathfrak{P}$ of the variable $p$, and of *functions* $\mu$ of the variable $p$ on the range $\mathfrak{P}$ recognized in current Analysis.

From the linear continuum with its infinite variety of functions and corresponding singularities G. CANTOR developed his *theory of classes of points* (*Punktmengenlehre*) with the notions: limit-point, derived class, closed class, perfect class, etc., and his *theory of classes in general* (*allgemeine Mengenlehre*) with the notions: cardinal number, ordinal number, order-type, etc. These theories of Cantor are permeating Modern Mathematics.† Thus there is a theory of functions on point-sets, in particular, on perfect point-sets, and on more general order-types, while the arithmetic of cardinal numbers and the algebra and function theory of ordinal numbers are under development.

Less technical generalizations or analogues of functions of the continuous real variable occur throughout the various doctrines and applications of Analysis. A function of several variables is a function of a single multipartite variable; a distribution of potential or a field of force is a function of position on a curve or surface or region; the value of the definite integral of the Calculus of Variations is a function of the variable function entering the definite integral; a curvilinear integral is a function of the path of integration; a functional operation is a function of the argument function or functions; etc., etc.

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* Cf. my paper *On a Form of General Analysis, with application to Linear Differential and Integral Equations*, read before the Section on Analysis of the Rome Congress of 1908, Atti, etc., vol. 2 (1909), pp. 98–114.

† Cf. A. SCHOENFLIES, *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten, Bericht erstattet der Deutschen Mathematiker-Vereinigung*, Jahresbericht, Band VIII (1900), Ergänzungsband II (1908). — A. SCHOENFLIES, *Mengenlehre*, Encyklopädie der mathematischen Wissenschaften, $I_1$, art. IA5 (1899), pp. 184–207. — R. BAIRE-A. SCHOENFLIES, *Theorie des ensembles*, Encyclopédie des sciences mathématiques, $I_1$, art. I 7 (1909), pp. 489–531.

A multipartite variable itself is a function of the variable index of the part. Thus a finite sequence: $x_1; \cdots; x_n$, of real numbers is a function $x$ of the index $i$, viz., $x(i) = x_i$ $(i = 1; \cdots; n)$. Similarly, an infinite sequence: $x_1; \cdots; x_n; \cdots$, of real numbers is a function $x$ of the index $n$, viz., $x(n) = x_n$ $(n = 1; 2; \cdots)$. Accordingly, $n$-fold algebra and the theory of sequences and of series are embraced in the theory of functions.

As apart from the determination and extension of notions and theories in analogy with simpler notions and theories, there is the extension by direct generalization. The Cantor movement is in this direction. Finite generalization, from the case $n = 1$ to the case $n = n$, occurs throughout Analysis, as, for instance, in the theory of functions of several independent variables. The theory of functions of a denumerable infinity of variables* is another step in this direction.

We notice a more general theory dating from the year 1906. Recognizing the fundamental rôle played by the notion *limit-element* (number, point, function, curve, etc.) in the various special doctrines, M. Fréchet has given, with extensive applications, an abstract generalization† of a considerable part of Cantor's theory of classes of points and of the theory of continuous functions on classes of points. Fréchet considers a general class $\mathfrak{P}$ of elements $p$ with the notion *limit* defined for sequences of elements. The nature of the elements $p$ is not specified; the notion *limit* is not explicitly defined; it is postulated as defined subject to specified conditions. For particular applica-

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[*] Cf. D. Hilbert, *Grundzuge einer allgemeine Theorie der linearen Integralgleichungen, vierte Mitteilung, funfte Mitteilung*, Gottinger Nachrichten, 1906, pp. 157–227, 439–480. *Wesen und Ziele einer Analysis der unendlich vielen unabhangigen Variabeln*, Rendiconti del Circolo Matematico di Palermo, vol. 27 (1909), pp. 59–74.

[†] M. Fréchet, *Sur quelques points du calcul fonctionnel*, Paris thesis, 1906; reprinted, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), pp. 1–74.

tions explicit definitions satisfying the conditions are given. More theorems are obtained by defining *limit* as usual in terms of the notion *distance* (*écart*, *voisinage*) postulated as defined, subject to specified conditions, for the pairs of elements.†

We conclude this review of current Analysis with the remark that the functions considered are either functions $\mu$ of variables $p$ of specified character or functions $\mu$ on ranges $\mathfrak{P}$ with postulated features: e. g., *limit; distance; element of condensation; connection*, of specified character.

In the form of General Analysis here in question we propose to consider *functions* $\mu$ *of a general variable* $p$ *on a general range* $\mathfrak{P}$. This *general* is the *true general*, embracing every well-defined particular case of variable and range. For the present we confine attention to *real- and single-valued functions* $\mu$.

A property of functions $\mu$ is said to be *of general reference* in case it is defined for functions $\mu$ of a general variable $p$ on a general range $\mathfrak{P}$. Thus *finiteness on the range* $\mathfrak{P}$ and *constancy on the range* $\mathfrak{P}$ are of general reference. On the other hand, e. g., the property: *continuity on the range* $\mathfrak{P}$, is *of special reference*, its definition being with reference to some special feature, e. g., *limit* or *distance*, postulated as defined for the range $\mathfrak{P}$.

I proceed to indicate certain properties of general reference to be considered in this memoir.

The function $\mu_1$ is dominated by the function $\mu_2$ in case $|\mu_1| \leqq |\mu_2|$, viz., for every value of $p$ $|\mu_1(p)| \leqq |\mu_2(p)|$.

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† I add two remarks.

A contribution to Fréchet's theory is made by T. H. Hildebrandt in his Chicago dissertation of 1910. The *distance function* $\delta(p_1, p_2)$ is replaced by a *relation* $K_{p_1 p_2 m}$, where $m$ is an integer, generalizing the relation: $\delta(p_1, p_2) \leqq 1/2^m$. This relation was suggested by the relation $K_{p_1 p_2 m}$ of Part II of this memoir. Mr. Hildebrandt further investigates the interrelations of the various conditions.

F. Rietz, in his paper: *Stetigkeitsbegriff und abstracte Mengenlehre*, read before the section on Analysis of the Rome Congress (Atti, etc., vol. 2 (1909), pp. 18–24), indicates a more general theory involving, instead of Fréchet's *limit* for sequences of elements, the notions: *element of condensation* (*Verdichtungsstelle*); *connection* (*Verkettung*), postulated as defined for subclasses of the class $\mathfrak{M}$; for pairs of subclasses of the class $\mathfrak{M}$.

A sequence $\{\mu_n\}$ of functions $\mu_n$ $(n = 1 ; 2 ; \cdots)$ may have the property that there exists a function $\mu$ and a sequence $\{a_n\}$ of real numbers $a_n$ $(n = 1 ; 2 ; \cdots)$ such that for every positive integer $n$ the function $\mu_n$ is dominated by the function $a_n \mu$, viz., for every $n$ and $p$ $|\mu_n(p)| \leqq |a_n \mu(p)|$.

Further, the *convergence on* $\mathfrak{P}$ of a sequence $\{\mu_n\}$ of functions to a function $\mu$, in notation:

$$\underset{n}{\mathrm{L}}\,\mu_n = \mu \quad (p);$$

the *uniform convergence* on $\mathfrak{P}$, in notation:

$$\underset{n}{\mathrm{L}}\,\mu_n = \mu \quad (\mathfrak{P});$$

and the *uniform convergence on* $\mathfrak{P}$ *relative to the function* $\sigma$ *as scale of uniformity*, in notation:

$$\underset{n}{\mathrm{L}}\,\mu_n = \mu \quad (\mathfrak{P} ; \sigma),$$

are properties of general reference. Here relatively uniform convergence[1] differs from uniform convergence only by the substitution, in the final inequality of the definition, for $e$ of $|e\,\sigma(p)|$, $e$ being the arbitrarily assigned positive number. Thus, uniformity of convergence is the special case of relative uniformity in which the scale function $\sigma$ is the constant function 1.

Since individual functions and pairs and sequences of functions are only special cases of classes of functions, we may

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[1] As an illustration of a proposition involving relative uniformity of convergence the following (cf. my Rome paper, loc. cit., p. 103) may serve.

The range $\mathfrak{P}$ being the class of all continuous functions $p(u)$ on the finite interval $\mathfrak{U}: a \leqq u \leqq b$, of the real number system, the necessary and sufficient condition that a sequence $\{p_n\}$ of continuous functions $p_n(u)$ converge uniformly on the interval $\mathfrak{U}$ is that the sequence $\{\mu_n\}$ of functions $\mu_n$:

$$\mu_n(p) = \int_a^b p_n(u)\,p(u)\,du,$$

of the variable continuous function $p$ converge on the range $\mathfrak{P}$ uniformly as to the scale function $\sigma$:

$$\sigma(p) = \int_a^b |p(u)|\,du.$$

In this example, the argument $p$ of the function $\mu$ being itself a function of the variable $u$, the function $\mu$ is, more precisely, a functional operation.

define General Analysis as *the theory of classes* $\mathfrak{M}$ *of functions* $\mu$ *of a general variable* $p$ *on a general range* $\mathfrak{P}$, and we consider properties of general reference of such classes $\mathfrak{M}$ of functions.

For instance, the class $\mathfrak{M}$ of all continuous functions $\mu$ on a finite interval $\mathfrak{P}$: $a \leqq p \leqq b$, of the real number system has the following properties of general reference:

1. Every function $\mu$ of the class $\mathfrak{M}$ is finite on the range $\mathfrak{P}$.
2. The class $\mathfrak{M}$ contains every function $\theta$ constant on the range $\mathfrak{P}$.
3. The class $\mathfrak{M}$ contains every function $\theta$ of the form:

$$\theta = a_1\mu_1 + a_2\mu_2,$$

where $\mu_1$, $\mu_2$ are functions of $\mathfrak{M}$ and $a_1$, $a_2$ are real numbers.

4. The class $\mathfrak{M}$ contains every function $\theta$ of the form:

$$\theta = \mathbf{L}_{n}\, \mu_n \quad (\mathfrak{P}),$$

viz., every function $\theta$ expressible as the limit of a sequence $\{\mu_n\}$ of functions $\mu_n$ $(n = 1;\ 2;\ \cdots)$ of the class $\mathfrak{M}$, the sequence converging uniformly on the range $\mathfrak{P}$.

Now the property 4 is *for this class* $\mathfrak{M}$ (and for any class $\mathfrak{M}$ having the properties 1, 2) equivalent to the property:

4′. The class $\mathfrak{M}$ contains every function $\theta$ of the form:

$$\theta = \mathbf{L}_{n}\, \mu_n \quad (\mathfrak{P};\ \mu),$$

viz., every function $\theta$ expressible as the limit of a sequence $\{\mu_n\}$ of functions of the class $\mathfrak{M}$, the sequence converging on the range $\mathfrak{P}$ uniformly as to a function $\mu$ likewise of the class $\mathfrak{M}$.

Again, the class $\mathfrak{M}$ of all absolutely convergent series has the properties 1, 3, 4′ but not the properties 2, 4. It has also the property:

5. For every sequence $\{\mu_n\}$ of functions of the class $\mathfrak{M}$ there exists a function $\mu$ of the class $\mathfrak{M}$ and a sequence $\{a_n\}$ of real numbers such that for every $n$ the function $\mu_n$ is dominated by the function $a_n\mu$.

The class $\mathfrak{M}$ of all continuous functions on the interval $\mathfrak{P}: a \leqq p \leqq b$, has also this property 5; indeed, the property 5 is implied by the properties 1, 2.

A class $\mathfrak{M}$ having the property: 3; 4′; 5, is said to be *linear;* to be *closed;* to have *the dominance property* $D$, or for brevity, to have the property:

$$L;\ C;\ D.$$

In this memoir I study in some detail certain closure and dominance properties of classes $\mathfrak{M}$ of functions. Of these properties the most important are the properties $L$, $C$, $D$. We speak of the *genus* ($LCD$) of all systems ($\mathfrak{P}$; $\mathfrak{M}$) whose classes $\mathfrak{M}$ have the composite property $LCD$. The property $D$ plays an important rôle in situations involving double limits in connection with relative uniformity of convergence, and relative uniformity enters into the definition of the special closure property $C$.

*Part I is devoted to the consideration of classes* $\mathfrak{M}$ *having a common range* $\mathfrak{P}$.—A class $\mathfrak{M}$ gives rise to the class $\mathfrak{M}_L$, *the class* $\mathfrak{M}$ *extended to be linear;* viz., the class consisting of all functions $\mu_L$ of the form:

$$\mu_L = a_1\mu_1 + \cdots + a_n\mu_n,$$

where $\mu_1; \cdots; \mu_n$ are functions of the class $\mathfrak{M}$ and $a_1; \cdots; a_n$ are real numbers. Two classes $\mathfrak{M}$, $\Xi$ give rise to a class $\mathfrak{M}_{\Xi}$, *the class* $\mathfrak{M}$ *extended as to the class* $\Xi$, consisting of all functions $\theta$ of the form:

$$\theta = \underset{n}{\mathrm{L}}\,\mu_n \quad (\mathfrak{P};\ \sigma),$$

viz., of all functions $\theta$ expressible as the limit of a sequence $\{\mu_n\}$ of functions $\mu_n$ ($n = 1; 2; \cdots$) of the class $\mathfrak{M}$, the sequence converging on the range $\mathfrak{P}$ uniformly as to some function $\sigma$ of the class $\Xi$. The *$*$-extension* $\mathfrak{M}_*$ *of the class* $\mathfrak{M}$ is the class $(\mathfrak{M}_L)_{\mathfrak{M}}$. We have the proposition:

$$\mathfrak{M}^D\ .\supset.\ \mathfrak{M}_*^{LCD},$$

viz., if a class $\mathfrak{M}$ has the dominance property $D$, then its $*$-extension $\mathfrak{M}_*$ is linear and closed and has the dominance property $D$.

*Part II is devoted to the consideration of classes $\mathfrak{N}$ of functions $\rho$ of two independent variables $p'$, $p''$ on the ranges $\mathfrak{P}'$, $\mathfrak{P}''$*, that is, of the bipartite variable $p = (p', p'')$ on the composite range $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$.—Of two classes $\mathfrak{M}'$ on $\mathfrak{P}'$ and $\mathfrak{M}''$ on $\mathfrak{P}''$ the $*$-*composite class** $(\mathfrak{M}'\mathfrak{M}'')_*$ is the $*$-extension of the class $\mathfrak{M}'\mathfrak{M}''$ of functions $\mu'\mu''$ on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ obtained by multiplying the various functions $\mu'$ of $\mathfrak{M}'$ by the various functions $\mu''$ of $\mathfrak{M}''$. The genus $(LCD)$ of systems $(\mathfrak{P}\,;\,\mathfrak{M})$ is closed under simultaneous composition of ranges $\mathfrak{P}$ and $*$-composition of classes $\mathfrak{M}$.

As of fundamental importance for theories involving functions of several variables, e. g., differential equations and integral equations, I raise the question of the functional characterization of the functions $\rho$ of the $*$-composite $(\mathfrak{M}'\mathfrak{M}'')_*$ of two given classes $\mathfrak{M}'$, $\mathfrak{M}''$. *The class $\mathfrak{M}''$ being an arbitrary class with the composite property $LCD$* $\big($i. e., of the genus $(LCD)\big)$, I secure such a characterization in case the class $\mathfrak{M}'$ is the class of continuous functions on the range $\mathfrak{P}' : a' \leqq p' \leqq b'$, or the class of absolutely convergent series on the range $\mathfrak{P}' : p' = 1\,;\,2\,;\,\cdots$, or more generally, any class having besides the dominance property $D$ certain properties:

$$K'_1\,;\;K'_2\,;\;\Delta',$$

defined in terms of a postulated

*development* $\Delta'$

of the range $\mathfrak{P}'$. This development $\Delta'$ of the general range $\mathfrak{P}$

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*For instance, the classes $\mathfrak{M}'$, $\mathfrak{M}''$ being the classes of continuous functions of the respective variables $p'$, $p''$ on two finite intervals $\mathfrak{P}'$, $\mathfrak{P}''$ the $*$-composite class $(\mathfrak{M}'\mathfrak{M}'')_*$ is the class of continuous functions of the bipartite variable $p = (p', p'')$ on the corresponding rectangle $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ of the bipartite number-system.

Again, the class $\mathfrak{M}'$ being the class of all functions of the positive integral variable $p' \equiv n$, viz., of all sequences of real numbers, and the class $\mathfrak{M}''$ being any class of genus $(LCD)$ on the range $\mathfrak{P}''$, the $*$-composite class $(\mathfrak{M}'\mathfrak{M}'')_*$ is in effect the class of all sequences $\{\mu''_n\}$ of functions $\mu''_n$ of the class $\mathfrak{M}''$. (Cf. § 65*d*.) Such a sequence of functions $\mu''_n$ quâ function $\phi(p', p'')$ is dominated by the product $\mu'(p')\mu''(p'')$ of certain two functions $\mu'$, $\mu''$ of the classes $\mathfrak{M}'$, $\mathfrak{M}''$, since $\mathfrak{M}''$ has the dominance property $D$.)

is a generalization, on the one hand, of the partition by sequential halving of the range $\mathfrak{P}' : a' \leq p' \leq b'$, and, on the other hand, of the denumeration ($p' = 1 ; 2 ; \cdots$) of the range $\mathfrak{P}' : p' = 1 ; 2 ; \cdots$.

Corresponding to the composition of ranges $\mathfrak{P}'$, $\mathfrak{P}''$ there is a composition of developments $\Delta'$, $\Delta''$. The genus ($LCDK_1K_2\Delta$) of systems ($\mathfrak{P} ; \Delta ; \mathfrak{M}$) whose classes $\mathfrak{M}$ have the composite property $LCDK_1K_2\Delta$ is closed under simultaneous composition of ranges $\mathfrak{P}$, composition of developments $\Delta$, and × composition of classes $\mathfrak{M}$.

I note that, for a properly defined development $\Delta$ of the range $\mathfrak{P}$, the composite property $LCDK_1K_2\Delta$ belongs, not only to the class of all continuous functions and to the class of all absolutely convergent series of real numbers, but to the class of all sequences of real numbers, and to the class of all sequences converging to null, and, $e$ being a positive number, to the class of all sequences $\mu$ such that the series:

$$|\mu(1)|^e + |\mu(2)|^e + \cdots,$$

is absolutely convergent. Accordingly the composite property belongs also to all the classes arising from these classes by repeated applications of the composition process indicated above.

The principal notions and theorems of this introductory memoir having now been indicated, we define General Analysis more precisely as *the theory of systems of classes of functions, functional operations, etc., involving at least one general variable on a general range.*

*Pure systems* of General Analysis involve only general variables and ranges; *mixed systems* involve besides general variables and ranges also special variables and ranges.

Thus, a general theory of Integral Equations may be framed * as a theory of the pure system:

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* As in my Colloquium Lectures, for the HILBERT analogies, following the development of E. SCHMIDT, *Entwickelung willkürlicher Funktionen nach Systemen vorgeschriebener*, Göttingen dissertation, 1905; reprinted, Mathematische Annalen, vol. 63 (1906), pp. 433–467.

$$(\mathfrak{P};\ \mathfrak{M};\ \mathfrak{K};\ J),$$

subject to specified conditions of general reference. Here $\mathfrak{M}$ is a class of functions of the general variable $p$ on the range $\mathfrak{P}$; $\mathfrak{K}$ is a class of functions of two independent variables $p_1, p_2$ on the range $\mathfrak{P}$; and $J$ is a functional operation transforming every pair $\mu_1, \mu_2$ of functions of the class $\mathfrak{M}$ into a real number $J(\mu_1, \mu_2)$.*

A *special system* of Analysis, that is, one involving only special variables and ranges, becomes a *mixed system* of General Analysis by the suitable adjunction of a class $\mathfrak{M}$ of functions of a general variable $p$ on a general range $\mathfrak{P}$. For example, from the system:

$$(\mathfrak{T};\ \mathfrak{C};\ K),$$

where $\mathfrak{C}$ is the class of continuous functions $\gamma$ of the real variable $t$ on the finite interval $\mathfrak{T}: a \leqq t \leqq b$, and $K$ is functional operation transforming a function of $\mathfrak{C}$ into a function of $\mathfrak{C}$, with its problem of the integration of the differential equation:

$$D\gamma = K\gamma \qquad \left(D = \frac{d}{dt}\right),$$

we secure by adjunction the system:

$$(\mathfrak{T};\ \mathfrak{C};\ \mathfrak{P};\ \mathfrak{M};\ K),$$

where $K$ is a functional operation transforming a function $\rho$ of the class $\mathfrak{R} = (\mathfrak{C}\mathfrak{M})_*$ into a function of the class $\mathfrak{R}$, with its problem of integration of the differential equation:

$$D\rho = K\rho \qquad \left(D = \frac{\partial}{\partial t}\right).$$

In case the range $\mathfrak{P}$ consists of a single element, we have the

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* For the simplest transcendental case $\mathfrak{M}$ is the class of continuous functions on the finite interval $\mathfrak{P}: a \leqq p \leqq b$, $\mathfrak{K}$ is the class of continuous functions of the independent variables $p_1, p_2$ on the interval $\mathfrak{P}$, and $J$ is the corresponding definite integral operation:

$$J(\mu_1, \mu_2) = \int_a^b \mu_1(p)\,\mu_2(p)\,dp.$$

original problem. In case $\mathfrak{P}$ consists of a finite number (denumerable infinity) of elements, we have the problem of integration of a finite (denumerably infinite) system of ordinary differential equations involving a finite (denumerably infinite) system of unknown continuous functions. In case $\mathfrak{P}$ is a finite interval $a' \leqq p \leqq b'$ of the real variable $p$, we have, for instance, the problem of integration of a differential-integral equation. In my Rome paper I gave conditions of general reference (as to the variable $p$ and the range $\mathfrak{P}$) ensuring the integrability of the general differential equation: $D\rho = K\rho$, $K$ being a linear homogeneous functional operation. In a paper to be presented to the London Mathematical Society I intend to simplify this solution and to remove from the functional operation $K$ the restriction of linear homogeneity.

In general, we adjoin the class $\mathfrak{M}$ to the special system with its known problem and theory in such a way that the mixed system with its problem are capable of specialization to the special system and problem, usually, as in the example just cited, in case the range $\mathfrak{P}$ consists of a single element. Then we impose upon the class $\mathfrak{M}$ and the other features of the mixed system conditions of general reference (as regards the variable $p$ and the range $\mathfrak{P}$) of such a nature as to validate a general theory of the problem for the mixed system capable of specialization to the special theory of the problem for the special system.

From special systems of Analysis pure systems of General Analysis arise, as in the case of Integral Equations, by the complete replacement of special variables and ranges by general variables and ranges; and mixed systems of General Analysis arise, not only by the process of adjunction just indicated, but also by the partial replacement of special variables and ranges by general variables and ranges. For example, the theory of functional characterization of the functions of the $*$-composite $(\mathfrak{M}'\mathfrak{M}'')_*$ of two classes $\mathfrak{M}'$, $\mathfrak{M}''$ of functions, outlined above and developed in Part II of this memoir, has one of its roots in the well-known proposition that the continuous functions $\rho$ of two real variables $p'$, $p''$ on the intervals $\mathfrak{P}' : a' \leqq p' \leqq b'$, $\mathfrak{P}'' : a'' \leqq p'' \leqq b'$ may be characterized as the functions $\rho$ which for every $p'$ are continuous functions of $p''$ on the interval $\mathfrak{P}''$ and which for every $p''$ are continuous functions of $p'$ on the interval $\mathfrak{P}'$ and accordingly are uniformly continuous functions on the interval $\mathfrak{P}'$, this uniform continuity on $\mathfrak{P}'$ being moreover uniform as to the parameter $p''$ on the interval $\mathfrak{P}''$.

With several remarks of special nature I conclude this Preface. The term *class* or *range* denotes a collection (*ensemble*, *Menge*) of unrepeated entities or elements well-defined individually and in their totality. This notion, fundamental for all Mathematics, is at present undergoing severe critical examination. Since mathematicians have reached no general consensus of opinion, I am content to use the term naively, as was usual before the recent outburst of criticism. However I make use of no transfinite argumentations.

For the sake of clearness and to avoid undue prolixity I have found it very convenient and almost necessary to make use of numerous technical and logical notations. In view of the definitions and repeated interpretations the reader should find these notations perfectly perspicuous. A list of logical notations is to be found at the end of the memoir.

As an alumnus and sometime instructor of Mathematics at Yale University I may express my pleasure in participating in the New Haven Colloquium and in the fact that this volume of the Colloquium Lectures on Mathematics is being published by Yale University.

# PART I.

## FUNDAMENTAL CLOSURE AND DOMINANCE PROPERTIES OF CLASSES OF FUNCTIONS OF A GENERAL VARIABLE.

### *Introduction.* § 1.

1. The *General Analysis* which we are approaching is *analysis*, in that it relates to the class $\mathfrak{A}$ of real numbers, and it is *general* in that it relates also to a class $\mathfrak{P}$ entering the theory in the fashion to be explained below.

It is postulated that the class $\mathfrak{P}$ has at least one element, and in this paper we consider in particular four cases. Case I: $\mathfrak{P}$ is *singular*, consisting of a single element. Case II: $\mathfrak{P}$ is *finite*, consisting of a finite number of elements; subcase $\mathrm{II}_n$: $\mathfrak{P}$ consists of $n$ elements. Case III: $\mathfrak{P}$ is *denumerably infinite*, consisting of a denumerable infinitude of elements. Case IV: $\mathfrak{P}$ consists of an infinitude of elements whose cardinal number is the cardinal number of the linear continuum. Here the cases I and $\mathrm{II}_1$ are identical, and the cases I, II, III constitute the case $\mathfrak{P}$ *denumerable*, while the case IV is an especially important subcase of the case $\mathfrak{P}$ *non denumerable.*

These four cases however are merely special cases to which the general theory applies.

In the general theory *the class $\mathfrak{P}$ of elements $p$ enters as the range $\mathfrak{P}$ of the argument $p$ of certain real and single valued functions $\mu$, constituting in their totality a class $\mathfrak{M}$ of such functions.* Thus, putting the class $\mathfrak{A}$ of real numbers into evidence, we may say elliptically that the theory relates to systems $\Sigma$ of the form:

$$\Sigma = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M}).$$

For special ranges $\mathfrak{P}$ we may as usual define properties of individual functions $\mu$. Thus, under case IV, we have the class $\mathfrak{M}$ of all functions $\mu$ continuous on the linear interval $0 \leqq p \leqq 1$. Again, under case III, we have the class $\mathfrak{M}$ of all absolutely convergent series. Such properties: *continuity; absolute convergence*, of functions are called *properties of special reference*, in that they are defined not for functions on an arbitrary range $\mathfrak{P}$ but only for functions on special ranges $\mathfrak{P}$. *Properties of general reference* are for instance these: *constancy; finiteness* (on the range $\mathfrak{P}$).

Similarly, for classes of functions we speak of properties of special and of general reference.

Part I is devoted to the consideration of the *interrelations of systems $\Sigma$ with a common arbitrary range $\mathfrak{P}$ whose classes $\mathfrak{M}$ of functions have certain properties of general reference.* These properties are certain *closure* and *dominance properties.*

For example, under case IV, the class $\mathfrak{M}$ of all continuous functions contains all functions of one of the form: ($A$) The absolute function $|\mu|$ of a function $\mu$ of the class $\mathfrak{M}$; ($L$) A linear homogeneous combination $a_1\mu_1 + a_2\mu_2$ with constant coefficients $a_1$, $a_2$ of two functions $\mu_1$, $\mu_2$ of the class $\mathfrak{M}$; ($C_0$) The limit of a uniformly convergent sequence $\{\mu_n\}$ of functions of the class $\mathfrak{M}$, the convergence being uniform on the range $\mathfrak{P}$.

Similarly, under case III, the class $\mathfrak{M}$ of all absolutely convergent series has the closure properties $A$, $L$, but not the property $C_0$ just defined.

However, introducing instead of uniformity of convergence the more general notion of *relative uniformity of convergence as to a scale function $\sigma$* (which, in case $\sigma$ is the constant function 1, is uniformity of convergence in the usual sense), we have a closure property $C$ common to the two classes, viz., the property of containing all functions of the form: ($C$) The limit of a sequence $\{\mu_n\}$ of functions of the class $\mathfrak{M}$, which converges on the range $\mathfrak{P}$ uniformly as to a scale function $\mu$ of the class $\mathfrak{M}$.

These *closure properties:*

$$A \equiv \textit{absolute}; \qquad L \equiv \textit{linear}; \qquad C \equiv \textit{closed},$$

are the principal ones studied. The principal question raised is that of the *extension of a class* $\mathfrak{M}$ *to a class* $\mathfrak{M}_P$ (containing $\mathfrak{M}$ and) *having the closure property* $P$. This question is solved for any combination of the closure properties: $A$; $L$; $C'$, by exhibition of explicit formulas (§§ 31, 43, 44), in case the class $\mathfrak{M}$ has the fundamental *dominance property* $D$, defined in § 19.

The propositions of the theory are for the most part statements of relations between various classes of functions possessing various properties of general reference. These relations are expressed in formulas comparable in compactness and clearness to the current formulas of analysis. The general mathematical and logical notations to be used in these formulas are introduced in §§ 2, 3. From the developments of the sequel the reader will appreciate the convenience of these notations, at least for the mathematical purposes of the present investigation.

*The four properties*: $A$; $L$; $C'$; $D$, of classes $\mathfrak{M}$ of functions are proved to be *completely independent* and *mutually consistent*, in the sense defined in § 17, by the exhibition (§ 18) of 16 classes $\mathfrak{M}$ having the four properties: $A$; $L$; $C'$; $D$ or their negatives: $\bar{A}$; $\bar{L}$; $\bar{C}'$; $\bar{D}$, in the various $2^4$ conceivable combinations:

$$ALC'D;\ ALC'\bar{D};\ \cdots;\ \bar{A}\,\bar{L}\,C'D;\ \bar{A}\,\bar{L}\,\bar{C}'\,\bar{D}.$$

## *Classes and properties of elements. General notations.* § 2.

2. $\mathfrak{P}$ or $[p]$ denotes *a class* (collection, aggregate, set, assemblage, *ensemble*, *Menge*) *of elements* $p$. The elements $p$ are supposed to be well-defined individually and in their totality.

In general, the bracket $[\ ]$ denotes *a class of elements*, the elements being indicated by the notation within the bracket. The elements of a class may be of any type, e. g., points, curves, collineations, numbers, functions, functional transformations, matrices, properties or pairs or triples or classes or sequences of elements of a subordinate type, etc., etc. *We denote elements of the various types occurring throughout the investigation by notations connoting the type.* Thus, single-valued, real-valued functions are denoted by Greek small letters, classes are denoted by German capital letters, and in particular, the classes of

*all real numbers;*
*all positive real numbers;*
*all positive rational integers,*

are denoted respectively by the various notations:

$$\mathfrak{A},\ [a],\ [b];$$
$$[e],\ [d];$$
$$[n],\ [m].$$

Accordingly, the definitions of convergence of a sequence and of continuity of a function have the respective forms: for every $e$ there exists an $n$ such that, etc.; for every $e$ there exists a $d$ such that, etc.

Discrimination between elements of the same class is usually made by means of suffixes; accordingly, $p$, $p_1$, $p_2$, $\cdots$ are elements of the class $\mathfrak{P}$. Discrimination between classes is usually made by superscripts; accordingly, $\mathfrak{P}' = [p']$, $\mathfrak{P}'' = [p'']$ are two classes. It is of course understood that elements (or classes) with distinct notations need not be distinct elements (or classes).

*An element $x$ belonging to a class $\mathfrak{P}$* is denoted by

$$x^{\mathfrak{P}};$$

thus the notations $p$, $p^{\mathfrak{P}}$ are interchangeable. *A class $\mathfrak{X} = [x]$ belonging to* or *contained in* or *a subclass of a class* $\mathfrak{P}$, that is, a class $\mathfrak{X}$ whose every element $x$ is an element of or belongs to the class $\mathfrak{P}$, is denoted by either notation:

$$\mathfrak{X}^{\mathfrak{P}},\ [x^{\mathfrak{P}}].$$

More generally, if $P$ denotes a property of elements, *an element $x$ having the property $P$* is denoted by

$$x^P,$$

and *a class $\mathfrak{X} = [x]$ of elements $x$ having individually the property $P$* is denoted by either notation:

$$\mathfrak{X}^P,\ [x^P].$$

Thus, with respect to a class $\mathfrak{P}$ of elements, the notation $\mathfrak{P}$ as denoting a property of elements denotes the property: *belonging to the class* $\mathfrak{P}$, and the notation $\mathfrak{P}$ as denoting a property of classes of elements denotes the property: *contained in* or *a subclass of the class* $\mathfrak{P}$.

*The class consisting of all elements p (of the class* $\mathfrak{P}$*) having the property P* is denoted by the notation:

$$[\text{all } p^P],$$

while the notation:

$$[p^P]$$

denotes *a class of elements p having the property P.** Thus the class $[\text{all } p^P]$ is a definite subclass of the class $\mathfrak{P}$ and contains every class $[p^P]$.

A class $[\mathfrak{P}]$ of classes $\mathfrak{P}$ of elements determines two classes:

$\mathsf{U}[\mathfrak{P}]$, *the least common superclass of the classes* $\mathfrak{P}$;
$\mathsf{\cap}[\mathfrak{P}]$, *the greatest common subclass of the classes* $\mathfrak{P}$.

$\mathsf{U}[\mathfrak{P}]$ is the class of all elements belonging to at least one class $\mathfrak{P}$ of the class $[\mathfrak{P}]$ of classes. $\cap[\mathfrak{P}]$ is the class of all elements belonging to every class $\mathfrak{P}$ of the class $[\mathfrak{P}]$ of classes.

The primary classes about which statements are made are supposed to exist, i. e., to have extension, to contain one or more elements. The derivative classes often do not exist, being the class containing no elements, the null-class. Thus, while the class $\mathsf{U}[\mathfrak{P}]$ always exists, the class $\cap[\mathfrak{P}]$ often does not exist.

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* These notations are with reference to a class $\mathfrak{P} = [p]$ and a property $P$ defined at least for elements $p$, e. g., by a single entry *table*, for every element $p$ the tabular entry being 1 or [illegible] according as the element has or has not the property.

Hence, the notation $[p]$ or $[p^{\mathfrak{P}}]$ denotes *the class of all elements p (belonging to* $\mathfrak{P}$), viz., has the meaning of the notation $[\text{all } p^{\mathfrak{P}}]$. Accordingly, the notation $[p]$ or $[p^{\mathfrak{P}}]$, having its meaning already determined, may not be used to denote *a class of elements p (belonging to* $\mathfrak{P}$), viz., *a subclass of* $\mathfrak{P}$. In practice, no confusion arises from this exception to the notational specifications of the text.

### *Properties of and propositions concerning elements. General notations.* § 3.

3. In Pure Mathematics *primary properties of elements* are directly defined, explicitly or by supposition, by means of a *table,* as explained in the footnote of § 2.

We consider the principal modes of *derivation of properties* from auxiliary properties.

*Negation of properties.* — From a property $P$ of elements arises by negation the property :

$$\bar{}P,$$

(read : *not P*), *the negative of the property P*, viz., the property of not having the property $P$.

*Conjunction or composition of properties.* — From $n$ properties :

$$P_1\,;P_2\,;\cdots;P_n,$$

of elements arises by composition the *composite property :*

$$P_1P_2\cdots P_n \quad \text{or} \quad P_1\,.\,P_2\,.\,\cdots\,.\,P_n,$$

(where the (.) is to be read *and*), viz., the property of having every property $P_i$ ($i = 1, 2, \cdots, n$). — More generally, from a class :

$$[P],$$

of properties of elements arises by composition the *composite property :*

$$\cap[P],$$

viz., the property of having every property $P$ of the class $[P]$ of properties. The notation is so chosen that, if for every property $P$ $\mathfrak{P}$ denotes the class of all elements having the property $P$, the class of all elements having the composite property $\cap[P]$ is $\cap[\mathfrak{P}]$, viz., the greatest common subclass of the classes $\mathfrak{P}$ of the resulting class $[\mathfrak{P}]$ of classes of elements.

*Disjunction of properties.* — From $n$ properties :

$$P_1;\ P_2;\ \cdots;\ P_n,$$

of elements arises by disjunction the *disjunctive property:*

$$P_1 \quad P_2 \cdots P_n,$$

(where the ( ) is to be read *or*), viz., the property of having at least one of the properties $P_i$ ($i = 1, 2, \cdots, n$).—More generally, from a class:

$$[P],$$

of properties of elements arises by disjunction the *disjunctive property:*

$$\cup[P],$$

viz., the property of having at least one of the properties $P$ of the class $[P]$ of properties. The notation is so chosen that the class of all elements having the disjunctive property $\cup[P]$ is the class $\cup[\mathfrak{P}]$, viz., the least common superclass of the classes $\mathfrak{P}$ of the corresponding class $[\mathfrak{P}]$ of classes of elements.

Composition and disjunction of the properties $P$ of a class:

$$(P_1; \cdots; P_n) \quad \text{or} \quad [P],$$

of properties are instances of the *propositional derivation* of properties from auxiliary properties. For we may write the definitions as follows:

$$x^{P_1 \cdots P_n} : \quad : x \ni i^{1 \leqq i \leqq n} . \supset . x^{P_i},$$

viz., an element $x$ having the composite property $P_1 \cdots P_n$ *is by definition* ( ) an element $x$ *such that* ($\ni$) for every integer $i$ having the property: $1 \leqq i \leqq n$, *it is true that* ($. \supset .$) $x$ has the property $P_i$;

$$x^{\cap P} : \quad : x \ni P^{P} . \supset . x^{P},$$

viz., an element $x$ has the composite property $\cap[P]$ in case for every property $P$ belonging to the class $[P]$ it is true that $x$ has the property $P$;

$$x^{P_1 \quad P_2 \quad P_n} . \quad . x \ni \exists\, i^{1 \leqq i \leqq n} \ni x^{P_i},$$

viz., an element $x$ has the disjunctive property $P_1 \cdots P_n$ in case *there exists an* ($\exists$) integer $i$ having the property: $1 \leqq i \leqq n$, such that $x$ has the property $P_i$;

$$x^{\cup P} . \quad . x \ni \exists\, P^{P} \ni x^{P},$$

viz., an element $x$ has the disjunctive property $\mathbf{U}\,[P]$ in case there exists a property $P$ belonging to the class $[P]$ such that $x$ has the property $P$.

Thus *composite and disjunctive properties are propositional*, and indeed respectively *implicational* and *existential.*

*Propositional derivation of properties.*—Types of *propositional properties** including not only composite and disjunctive properties, but perhaps all the propositional properties met with in mathematics are defined as follows:

A property $P$ of elements $x$ of a class:

$$\mathfrak{X} \equiv [x],$$

will be defined propositionally by the mediation of a class:

$$\mathfrak{Y} \equiv [y],$$

of elements $y$ and two properties:

$$Q;\ R,$$

of pairs $(x, y)$ of elements $x$, $y$.

A pair $(x, y)$ having the property $Q$ we indicate by the various notations:

$$(x, y)^Q;\ x^{Q_y};\ y^{Q_x},$$

the last two notations indicating that the property may be thought of as a property $Q_y$ (associated with the element $y$) of the element $x$, and likewise as a property $Q_x$ (associated with the element $x$) of the element $y$. The three notations are understood to have the same meaning; however, in case the classes $\mathfrak{X}$, $\mathfrak{Y}$ have elements in common, the notations $Q_x$, $Q_y$ should not be used simultaneously. For the present purpose, the logical emphasis is placed desirably if we use the properties $Q$, $R$ in the forms: $Q_x$, $R_y$.

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* Propositional properties are what RUSSELL calls *propositional functions*, the variable of the function being the element having or not having the property. However, the variable of RUSSELL is any term or element, while for practical mathematical purposes we suppose that it is an element of a certain class.

*Implicational property* $P$.—An element $x$ (of the class $\mathfrak{X}$) has the implicational property $P$ (derived from the class $\mathfrak{Y}$ and the properties $Q$, $R$) in case

$$y^{Q_x} . \supset . x^{R_y},$$

viz., for every $y$ (of the class $\mathfrak{Y}$) having the property $Q_x$ it is true that the element $x$ has the property $R_y$.

*Existential property* $P$.—An element $x$ (of the class $\mathfrak{X}$) has the existential property $P$ (derived from the class $\mathfrak{Y}$ and the properties $Q$, $R$) in case

$$\exists\ y^{Q_x} \ni x^{R_y},$$

viz., there exists an element $y$ (of the class $\mathfrak{Y}$) having the property $Q_x$ such that the element $x$ has the property $R_y$.

*For a class* $\mathfrak{X}$ *implicational and existential properties are interchanged by negation*, for the negative of the $\genfrac{}{}{0pt}{}{\text{implicational}}{\text{existential}}$ property derived from the class $\mathfrak{Y}$ and the properties $Q$, $R$ is the $\genfrac{}{}{0pt}{}{\text{existential}}{\text{implicational}}$ property derived from the class $\mathfrak{Y}$ and the properties $Q$, $^-R$. Further, the existential property is symmetric in the properties $Q$ and $R$, while the implicational property is unchanged if we replace $Q$, $R$ by $^-R$, $^-Q$.

*Examples.* — The composite and the disjunctive properties defined above correspond to the specifications :

$$\mathfrak{Y} \equiv [y] \equiv [i = 1, 2, \cdots, n];$$

$$y^{Q_x} \equiv y^{\mathfrak{Y}} \equiv i^{1 < i < n}; \qquad x^{R_y} \equiv x^{R_i} \equiv x^{P_i},$$

or, more generally,

$$\mathfrak{Y} \equiv [y] \equiv [P];$$

$$y^{Q_x} \equiv y^{\mathfrak{Y}} \equiv P^{[P]}; \qquad x^{R_y} \equiv x^{R_P} \equiv x^{P}.$$

Further, consider the convergence of a sequence :

$$\{a_n\} \qquad \text{or} \qquad a_1, a_2, \cdots, a_n, \cdots,$$

of numbers to a number $a$ as limit. This property of the sequence-individual $(\{a_n\}, a)$ has the definition :

$$e : \supset : \exists\ m \ni n > m . \supset . A(a_n - a) \leqq e,$$

viz., for every positive number $e$ it is true that there exists a positive integer $m$ such that for every positive integer $n > m$

it is true that the absolute value* of $a_n - a$ is less than or equal to $e$.

Thus, for the class $\mathfrak{X}$ of all sequence-individuals $(\{a_n\}, a)$, the property $P$: *convergence of the sequence* $\{a_n\}$ *of real numbers to the individual number* $a$ *as a limit,* is the implicational property derived from the class $\mathfrak{Y} \equiv [y] \equiv [e]$ and the property $Q_x$: *belonging to* $\mathfrak{Y}$, with a certain property $R_y \equiv R_e$. This property $R_e$ is the existential property derived from the class $\mathfrak{Y}'_e \equiv [y'_e] \equiv [m]$ and the property $Q'_{ex}$: *belonging to* $\mathfrak{Y}'_e$, with a certain property $R'_{ey'_e} \equiv R'_{em}$. This property $R'_{em}$ is the implicational property derived from the class $\mathfrak{Y}''_{em} \equiv [y''_{em}] \equiv [n]$ and the property $Q''_{emx}$: *greater than* $m$, with the property $R''_{emy_{em}''} \equiv R''_{emn}$ : $A(a_n - a) \leqq e$.

The negative of the convergence has the form:

$$\exists\ e \ni m \,.\supset.\, \exists\ n > m \ni A(a_n - a) > e,$$

viz., there exists a positive number $e$ such that for every positive integer $m$ it is true that there exists a positive integer $n > m$ such that the absolute value of $a_n - a$ exceeds $e$.

*Propositions concerning elements.*—The propositional properties of elements $x$ of the class $\mathfrak{X}$ defined above are propositions concerning the elements. Such a proposition involves as constants the element $x$, the class $\mathfrak{Y}$, and the properties $Q$ and $R$.

In the examples cited the property $Q$ of the pair $(x, y)$ is independent of $x$, that is, for an element $y$ the property $Q$ either does or does not hold for the pair $(x, y)$ whatever be the element $x$; in such a case the property $Q$ is a property of the element $y$, so that we may replace the notation $Q_x$ by $Q$, or, more precisely, the notation $y^{Q_x}$ or $(x, y)^Q$ by $x \,.\, y^Q$. Here $x$ quâ constant may be omitted.

*Special cases of propositional properties.*—It is logically instructive to consider the four special cases of implicational and existential properties derived from properties $Q$ and $R$ each independent either of $x$ or of $y$. The definitions are given in the following table:

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* For a real number $a$ the notation $Aa$ denotes the absolute value $|a|$ of $a$.

| Implicational $P$<br>Existential $P$ | $R$ on $\mathfrak{Y}$ | $R$ on $\mathfrak{X}$ |
|---|---|---|
| $Q$ on $\mathfrak{Y}$ | $y^Q . \supset . y^R$ | $y^Q . \supset . x^R$ |
|  | $\mathcal{H}\ y^Q \ni y^R$ | $\mathcal{H}\ y^Q \ni x^R$ |
| $Q$ on $\mathfrak{X}$ | $x^Q . y . \supset . y^R$ | $x^Q . y . \supset . x^R$ |
|  | $\mathcal{H}\ (x^Q, y) \ni y^R$ | $\mathcal{H}\ (x^Q, y) \ni x^R$ |

Here "$Q$ on $\mathfrak{Y}$" means that $Q$ being independent of $x$ is a property of elements $y$ of the class $\mathfrak{Y}$.

The first entry is read as follows. An element $x$ of the class $\mathfrak{X}$ has the implicational property $P$ (derived from the class $\mathfrak{Y}$) and the properties $Q$ on $\mathfrak{Y}$ and $R$ on $\mathfrak{Y}$) in case every element $y$ (of the class $\mathfrak{Y}$) having the property $Q$ has the property $R$.* This property $P$ is independent of $x$. It holds for $\genfrac{}{}{0pt}{}{\text{every}}{\text{no}}$ element $x$ in case there exists $\genfrac{}{}{0pt}{}{\text{no}}{\text{an}}$ element $y^{Q^-R}$: having the property $Q$ but not the property $R$. We may write this result as follows:

$$P = 1 \quad (Q{}^-R = 0);$$
$$P = 0 \quad (Q{}^-R > 0),$$

where $P = {}^1_0$ indicates that the property $P$ holds (under the adjoined conditions on the properties $Q$, $R$) for $\genfrac{}{}{0pt}{}{\text{every}}{\text{no}}$ element $x$, and $Q{}^-R \genfrac{}{}{0pt}{}{=}{>} {}^0_0$ indicates that the composite property $Q{}^-R$ (of $Q$ and ${}^-R$) holds for $\genfrac{}{}{0pt}{}{\text{no}}{\text{some}}$ element $y$.

The eight properties in reduced form are given in the following table:

|  |  |
|---|---|
| $P = 1 \quad (Q{}^-R = 0)$<br>$P = 0 \quad (Q{}^-R > 0)$ | $P = 1 \quad (Q = 0)$<br>$P = R \quad (Q > 0)$ |
| $P = 1 \quad (QR > 0)$<br>$P = 0 \quad (QR = 0)$ | $P = R \quad (Q > 0)$<br>$P = 0 \quad (Q = 0)$ |
| $P = 1 \quad (R = 1)$<br>$P = {}^-Q \quad (R < 1)$ | $P = {}^-Q \,{}^\smile R$ |
| $P = Q \quad (R > 0)$<br>$P = 0 \quad (R = 0)$ | $P = QR$ |

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* Or: "for every element $y$ (of the class $\mathfrak{Y}$) having the property $Q$ it is true that it has the property $R$."

Here, for instance, the entry: $P = {}^-Q \quad (R \prec 1)$, $Q$ bein $\mathfrak{X}$ and $R$ on $\mathfrak{Y}$), means that $P$ holds precisely for thos ments $x$ for which $Q$ does not hold, if $R$ holds not for element $y$, i. e., if ${}^-R$ holds for some element $y$; ${}^-R \succ$ ( $R \prec 1$ being equivalent conditions on $R$.

This table may be variously checked. First, as we above, the $\substack{\text{implicational}\\\text{existential}}$ property from $\mathfrak{Y}$; $Q$; $R$ is equivale the $\substack{\text{implicational}\\\text{existential}}$ property from $\mathfrak{Y}$; $\substack{^-R;\ Q\\R;\ Q}$, and the negative $\substack{\text{implicational}\\\text{existential}}$ property from $\mathfrak{Y}$; $Q$; $R$ is the $\substack{\text{existential}\\\text{implicational}}$ pro from $\mathfrak{Y}$; $Q$; ${}^-R$. Again, the case: $R$ independent of of $y$, viz., $R = 1$ or $R = 0$, is a subcase common to the $R$ on $\mathfrak{Y}$; $R$ on $\mathfrak{X}$, and similarly for $Q = 1$ or $Q = 0$.

### *Functions on the class $\mathfrak{P}$ to the class $\mathfrak{A}$.* §§ 4, 5.

4. A (single-valued) *function $F$ on the class $\mathfrak{P}$ to the* $\mathfrak{P}'$ is a correspondence $F$ of all elements $p$ to (all or n some) elements $p'$ such that for every element $p$ of $\mathfrak{P}$ ther definite corresponding element of $\mathfrak{P}'$, in notation: $F(p)$ or $F_p$. This *function $F$ on $\mathfrak{P}$ to $\mathfrak{P}'$* is G. CANTOR's *Bel von $\mathfrak{P}$ mit $\mathfrak{P}'$*; Mathematische Annalen, vol. 46 (1 p. 486.

It is convenient to adopt the terminology of the the functions. Thus, $p$ is the *variable* of the function $F$; th $\mathfrak{P}$ is the *range* (of the variable) of the function $F$; at the *ment value* or *place* $p$ the function $F$ has the *functional* $F_p$, etc.

We indicate such a function $F$ by the various notation

$$F,\ (F_p),\ (F_p \mid p),\ (F_p \mid p^{\mathfrak{P}}),$$

where the class $\mathfrak{P}'$ is to be understood from the context. last notation may be read: "the system of (functional) $F_p$ where $p$ varies over the class $\mathfrak{P}$." *The intention is t criminate sharply between function and functional value*

In view of the generality of $\mathfrak{P}$ the preceding explan cover the case of functions of two or more variables. T function:

$$F \equiv (F_{p'p''}) \equiv (F_{p'p''} \mid p'p'') \equiv (F_{p'p''} \mid p'^{\mathfrak{P}'} p''^{\mathfrak{P}''}),$$

of the two variables $p'$, $p''$ which vary independently over the classes $\mathfrak{P}'$, $\mathfrak{P}''$ (which may be identical) is a function:

$$F \equiv (F_p) \equiv (F_p \mid p) \equiv (F_p \mid p^{\mathfrak{P}}),$$

of the variable $p$ which ranges over the class $\mathfrak{P}$, the element $p$ being the bipartite complex $p'p''$ or $(p', p'')$ of the elements $p'$, $p''$ and the class $\mathfrak{P}$ being the class of such bipartite elements $p'p''$, in notation:

$$\mathfrak{P}'\mathfrak{P}'' \equiv [p'p''].$$

Such a function $F$ on $\mathfrak{P}'\mathfrak{P}''$ to some class $\mathfrak{P}'''$ is likewise a function $F$ on $\mathfrak{P}'$ to a certain class $\mathfrak{M}''$ of functions $M''$ on $\mathfrak{P}''$ to $\mathfrak{P}'''$, viz., the class $\mathfrak{M}''$ of all functions:

$$M'' \equiv F_{p'} \equiv (F_{p'p''} \mid p''^{\mathfrak{P}''}),$$

where $p'$ varies over $\mathfrak{P}'$, in notation:

$$\mathfrak{M}'' \equiv [\text{all } (F_{p'p''} \mid p''^{\mathfrak{P}''}) \mid p'^{\mathfrak{P}'}].$$

Conversely, a function $F$ on $\mathfrak{P}'$ to a class $\mathfrak{M}''$ of functions on $\mathfrak{P}''$ to $\mathfrak{P}'''$ is a function $F$ on $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{P}'''$.

5. We are at present concerned with *real-valued single-valued functions*, that is, functions on $\mathfrak{P}$ to the class $\mathfrak{A}$ of all real numbers. We denote these functions by small Greek letters: $\phi$, $\mu$, etc. The theory of functions on $\mathfrak{P}$ to $\mathfrak{A}$ may naturally precede the theory of functions on $\mathfrak{P}$ to more general classes $\mathfrak{P}'$. In particular, in accordance with the remarks of the preceding paragraph, one readily extends many of the notions and propositions of the sequel to the theory of functions on $\mathfrak{P}$ to classes $\mathfrak{P}'$ of functions on $\mathfrak{P}''$ to $\mathfrak{A}$, or to the theory of functions on $\mathfrak{P}$ to classes $\mathfrak{P}'$ of functions on $\mathfrak{P}''$ to classes $\mathfrak{P}'''$ of functions on $\mathfrak{P}''''$ to $\mathfrak{A}$, etc.; this holds true, for instance, of the fundamental notion of *relative uniformity of convergence* (§ 6).

*The class $\mathfrak{P}$ enters the theory by the mediation of a class $\mathfrak{M}$*:

$$\mathfrak{M} \equiv [\mu],$$

*of functions $\mu$ on $\mathfrak{P}$ to $\mathfrak{A}$*, so that the theory relates to the system $\Sigma \equiv (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M})$.

We notice in particular the following instances or cas the general system $\sum$:

(I) $\mathfrak{P}^{\mathrm{I}}$ is the class of a single element: e. g., $p = 1$.
$\mathfrak{M}^{\mathrm{I}}$ is the class of all functions on $\mathfrak{P}^{\mathrm{I}}$ to $\mathfrak{A}$,
in effect, $\mathfrak{M}^{\mathrm{I}}$ is the real number system $\mathfrak{A}$.

($\mathrm{II}_n$) $\mathfrak{P}^{\mathrm{II}_n}$ is the class of $n$ elements: e. g., $p = 1, 2,$ ·
$\mathfrak{M}^{\mathrm{II}_n}$ is the class of all functions $\mu$ on $\mathfrak{P}^{\mathrm{II}_n}$ to $\mathfrak{A}$,
of all $n$-partite real numbers:

$$\mu \equiv (\mu_p | p) \equiv (\mu_1, \mu_2, \cdots, \mu_n).$$

(III) $\mathfrak{P}^{\mathrm{III}}$ is the class of a denumerable infinitude o
ments: e. g., $p = 1, 2, \cdots, n, \cdots$.
$\mathfrak{M}^{\mathrm{III}}$ is the class of all functions $\mu$ on $\mathfrak{P}^{\mathrm{III}}$ to $\mathfrak{A}$,
of all infinite sequences of real numbers:

$$\mu = (\mu_p | p) = (\mu_1, \mu_2, \cdots, \mu_n, \cdots).$$

($\mathrm{III}_0$) $\mathfrak{M}^{\mathrm{III}_0}$ is the subclass consisting of all functions $\mu$ suc

$$\mathop{\mathbf{L}}_{p=\infty} \mu_p = 0.$$

($\mathrm{III}_e$) $\mathfrak{M}^{\mathrm{III}_e}$ is the subclass of all functions $\mu$ such that

$$\sum_{p=1}^{\infty} A\mu_p^e$$

converges.*

(IV) $\mathfrak{P}^{\mathrm{IV}}$ is the finite interval† (01):

$$0 \leqq p \leqq 1,$$

of the real number system $\mathfrak{A}$.
$\mathfrak{M}^{\mathrm{IV}}$ is the class of all continuous functions of
$\mathfrak{P}^{\mathrm{IV}}$ to $\mathfrak{A}$.

Thus, theory I is the theory of the real number syst one-dimensional analytic geometry; theory $\mathrm{II}_n$ is $n$-dimen analytic geometry; theory $\mathrm{III}_0$ is the theory of num sequences tending to the limit 0; theory $\mathrm{III}_1$ is the the absolutely convergent series; theory IV is the theory of valued single-valued continuous functions on the finite interval (01). The general theory of the sequel, duly particularized, belongs to each of these particular theories.

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* Here, as in the sequel, $A$ denotes *the absolute value of*; that is, fo real or complex number $a$, $Aa \equiv |a|$. Further, $e$ is a positive number. is a subclass of $\mathfrak{M}^{\mathrm{III}_0}$, and $e_1 < e_2$ implies that $\mathfrak{M}^{\mathrm{III}_{e_1}}$ is a subclass of $\mathfrak{M}$

† The theory relates equally to case IV for any interval $(a_0 a_1)$.

A function $\mu$ on $\mathfrak{P}^{\mathrm{II}_n}$ is a *point* or *vector* in $n$ dimensions expressed analytically by its $n$ coördinates associated with $n$ independent directions and units. Thus a value of $p$ specifies one direction and associated unit. Any class (*Menge*) $\mathfrak{M}$ of objects $\mu$ each capable of characterization or specification by means of $n$ measurements to scale may be represented as a class $\mathfrak{M}$ of functions $\mu$ on $\mathfrak{P}^{\mathrm{II}_n}$ or as a point-set in space $\mathfrak{M}^{\mathrm{II}_n}$ of $n$ dimensions.

We are at present interested in classes $\mathfrak{M} \equiv [\mu]$ possessing various combinations of properties held in common by the classes:

$$\mathfrak{M}^{\mathrm{I}}\,;\ \mathfrak{M}^{\mathrm{II}_n}\,;\ \mathfrak{M}^{\mathrm{III}}\,;\ \mathfrak{M}^{\mathrm{III}_0}\,;\ \mathfrak{M}^{\mathrm{III}_c};\ \mathfrak{M}^{\mathrm{IV}},$$

and in fact by many other essentially distinct classes of which some will be indicated below.[*] These properties involve a convenient *generalization of the important notion of uniformity*—the notion of *relative uniformity.*

*Relative uniformity. Range and scale of uniformity.* §§ 6–13.

6. If the notation

$$R_p$$

denotes the statement that a relation involving an element $p$ of the class $\mathfrak{P}$ holds for the element $p$, we denote by

$$R_p \quad (p)$$

the statement that the relation holds for every element $p$. On the basis of convenient definitions of (absolute) uniformity and of relative uniformity of validity of the relation in question, we denote by

$$R_p \quad (\mathfrak{P})$$

the statement that the relation holds *uniformly* on the *range* $\mathfrak{P}$, and by

$$R_p \quad (\mathfrak{P}\,;\ \sigma)$$

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[*] Cf. §§ 55 IV ; 74 I, II ; 84.

the statement that the relation holds *uniformly* on the *range* $\mathfrak{P}$ with respect to the function $\sigma$ as a *scale of uniformity.*

With respect to the relation in question the notions of uniformity on range $\mathfrak{P}$ and of relative uniformity on range $\mathfrak{P}$ as to scale function $\sigma$ are subject to definition, usually in such a manner that $R_p$ ($\mathfrak{P}$) implies $R_p$ ($p$) and that $R_p$ ($\mathfrak{P}$) and $R_p$ ($\mathfrak{P}$; 1) are equivalent. Here 1 denotes the (constant) function $\sigma$ having for every $p$ the functional value $\sigma_p = 1$.

For instance, with respect to a function $\phi$ on $\mathfrak{P}$ to $\mathfrak{A}$, we consider the relation $R_p$ that $\phi_p$ belongs to $\mathfrak{A}$, that is,

$$R_p \equiv \phi_p^{\mathfrak{A}}.$$

Then, by definition of $\phi$,

$$\phi_p^{\mathfrak{A}} \quad (p).$$

We say that the relation $\phi_p^{\mathfrak{A}}$ holds uniformly on $\mathfrak{P}$ in case the functional value $\phi_p$ is constant (independent of $p$), that is, the two statements:

$$\phi_p^{\mathfrak{A}} \quad (\mathfrak{P}); \qquad \phi_{p_1} = \phi_{p_2} \quad (p_1 p_2),$$

are equivalent.

We denote by $a(\mathfrak{P})$ *the constant* $a$ *quâ function of* $p$, and by $\mathfrak{A}(\mathfrak{P})$ *the class of such constant functions on* $\mathfrak{P}$ *to* $\mathfrak{A}$. Thus the statements:

$$\phi_p^{\mathfrak{A}} \quad (\mathfrak{P}); \qquad \phi^{\mathfrak{A}(\mathfrak{P})},$$

are equivalent.

Denoting by $\mathfrak{A}\sigma$ *the class of all numerical multiples* $a\sigma$ *of a function* $\sigma$: $\mathfrak{A}\sigma \equiv [a\sigma \mid a]$, we might define relative uniformity for the relation $\phi_p^{\mathfrak{A}}$ as follows:

$$\phi_p^{\mathfrak{A}} \quad (\mathfrak{P}; \sigma) \ \equiv \ \phi^{\mathfrak{A}\sigma}.$$

From a numerical relation:

$$R(a_1, a_2, \cdots),$$

involving numbers $a_1$, $a_2$, $\cdots$ we derive a relation:

$$R(\phi_{1p}, \phi_{2p}, \cdots),$$

involving an element $p$ of $\mathfrak{P}$ and functions $\phi_1$, $\phi_2$, $\cdots$ on $\mathfrak{P}$ to $\mathfrak{A}$, by substituting for the numbers $a_1$, $a_2$, $\cdots$ the functional values $\phi_{1p}, \phi_{2p}, \cdots$ for the argument $p$ of the functions $\phi_1, \phi_2, \cdots$. Further we derive a functional relation:*

$$R(\phi_1, \phi_2, \cdots),$$

involving functions $\phi_1, \phi_2, \cdots$ by the understanding that

$$R(\phi_1, \phi_2, \cdots) \equiv R(\phi_{1p}, \phi_{2p}, \cdots)\ (p);$$

that is, elliptically, *numerical relations amongst functions are understood to hold identically in the variables of the functions*, the context indicating the variables in which the respective functions are to be taken. Thus, $\phi \leqq \psi$ means $\phi_p \leqq \psi_p\ (p)$, and $\phi = 1$ means $\phi_p = 1\ (p)$, i. e., $\phi = 1(\mathfrak{P})$.

7. *Relatively uniform convergence.*—In this paper *relative uniformity* enters in the first place as *mode of convergence of sequences, whose terms are functions of the variable or parameter* $p$, and for this case the notion is elucidated in some detail.

7*a*. The sequence:

$$\{a_n\} \quad \text{or} \quad \{a_n | n\} \quad \text{or} \quad a_n \quad (n = 1, 2, \cdots),$$

of real numbers $a_n$ has as limit the number $a$: in notation,†

(1) $$\underset{n}{\mathbf{L}}\, a_n = a,$$

in case

(2) $$e : \supset : \exists\ n_e \ni n > n_e \,.\supset.\, A(a_n - a) \leqq e,$$

viz., for every positive number $e$ (it is true that) there exists a positive integer $n_e$ (dependent on $e$) of such a nature that for every positive integer $n$ exceeding $n_e$ (it is true that) the absolute value of $a - a_n$ is at most $e$.

For a fixed $e$ there is a least integer effective as $n_e$.

7*b*. The sequence $\{\mu_n\}$ of functions $\mu_n$ on $\mathfrak{P}$ to $\mathfrak{A}$ converges to the function $\theta$ as limit:

(1) $$\underset{n}{\mathbf{L}}\, \mu_n = \theta,$$

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* Derivation of functional relations from numerical relations is not commutative with negation.—The signs: $\equiv$; $\not\equiv$, denote primarily logical identity; diversity. Thus, $\phi \not\equiv \psi$ is the negative of $\phi \equiv \psi$, i. e., $\phi_p \equiv \psi_p\ (p)$, and not the functional relation derived from the numerical relation: $a \not\equiv b$.

† $\underset{n}{\mathbf{L}}$ denotes always $\underset{n \doteq \infty}{\mathbf{L}}$.

that is, in accordance with § 6,

(1′) $$\mathop{\mathrm{L}}_{n} \mu_{np} = \theta_p \quad (p),$$

in case

(2) $$p \,.\, e : \supset : \exists \; n_{pe} \ni n > n_{pe} \,.\, \supset .\, A(\mu_{np} - \theta_p) \leqq e,$$

viz., for every $p$ and $e$ (it is true that) there exists a positive integer $n_{pe}$ (dependent on $p$ and $e$) such that for every $n$ exceeding $n_{pe}$ (it is true that) the absolute value of $\mu_{np} - \theta_p$ is at most $e$.

For fixed $p$ and $e$ there is a least integer effective as $n_{pe}$.

For a fixed $e$ the system of least integers $n_{pe}$ where $p$ varies over $\mathfrak{P}$ may be finite (bounded from $\infty$), that is, there may exist an integer $n_e$ dependent on $e$ such that $n_{pe} \leqq n_e \; (p)$. If this is the case for every $e$, the convergence for every $p$ of $\mathfrak{P}$ holds uniformly on $\mathfrak{P}$. Cf. § 7*c*.

7*c*. The convergence holds uniformly on $\mathfrak{P}$:

(1) $$\mathop{\mathrm{L}}_{n} \mu_n = \theta \quad (\mathfrak{P}),$$

that is,

(1′) $$\mathop{\mathrm{L}}_{n} \mu_{np} = \theta_p \quad (\mathfrak{P}),$$

in case

(2) $$e : \supset : \exists \; n_e \ni n > n_e \,.\, \supset .\, A(\mu_n - \theta) \leqq e,$$

that is,

(2′) $$e : \supset : \exists \; n_e \ni n > n_e \,.\, \supset .\, A(\mu_{np} - \theta_p) \leqq e \quad (p),$$

viz., for every $e$ there exists a positive integer $n_e$ such that for every $n$ exceeding $n_e$ the absolute of the function $\mu_n - \theta$ is for every $p$ at most $e$.

Here, for a function $\sigma$, *the absolute of* $\sigma$, in notation: $A\sigma$, is the function on $\mathfrak{P}$ to $\mathfrak{A}$ for which $(A\sigma)_p = A(\sigma_p)$ for every $p$.

7*d*. $\sigma$ being a function on $\mathfrak{P}$ to $\mathfrak{A}$, the convergence holds uniformly on the range $\mathfrak{P}$ relatively to or as to the scale $\sigma$:

(1) $$\mathop{\mathrm{L}}_{n} \mu_n = \theta \quad (\mathfrak{P} ;\, \sigma),$$

in case

(2) $$e : \supset : \exists \; n_e \ni n > n_e \,.\, \supset .\, A(\mu_n - \theta) \leqq eA\sigma,$$

that is,

$$(2') \quad e : \supset : \exists\; n_e \ni n > n_e \,.\supset .\, A(\mu_{np} - \theta_p) \leqq eA\sigma_p \;\; (p),$$

viz., for every $e$ there exists a positive integer $n_e$ such that for every $n$ exceeding $n_e$ it is true that for every $p$ the absolute value of $\mu_{np} - \theta_p$ is at most $e$ multiplied by the absolute value of $\sigma_p$.

Thus, the definition (§ 7*c*2) of convergence uniform on $\mathfrak{P}$ becomes the definition (§ 7*d*2) of convergence on $\mathfrak{P}$ uniform as to a scale function $\sigma$ by the substitution of $eA\sigma$ for $e$ in the final inequality. Accordingly, uniform convergence is an instance of relatively uniform convergence, the scale function being the constant function 1.

In the formulas of absolutely or of relatively uniform convergence the elements $p$ may notationally be suppressed, since they enter similarly; cf. § 6. Thus the formulas in the functions alone have essentially the simplicity of corresponding formulas for convergent sequences of numbers.

7*e*. We notice the propositions:

1) For every $\sigma$ it is true that the relation:

$$\underset{n}{\mathrm{L}}\,\mu_n = \theta \quad (\mathfrak{P};\ \sigma),$$

implies the relation:

$$\underset{n}{\mathrm{L}}\,\mu_n = \theta \quad (p),$$

and indeed in such wise that, if at an element $p$ the scale $\sigma$ vanishes: $\sigma_p = 0$, the corresponding sequence $\{\mu_{np}\}$ of functional values $\mu_{np}$ are ultimately all equal to the corresponding functional value $\theta_p$ of the limit function $\theta$.

2) The relations:

$$\underset{n}{\mathrm{L}}\,\mu_n = \theta \quad (\mathfrak{P}); \qquad \underset{n}{\mathrm{L}}\,\mu_n = \theta \quad (\mathfrak{P};\ 1),$$

are equivalent.

7*f*. *Examples.**—The cases $\mathrm{III}_0$; $\mathrm{III}_1$; IV furnish illustrations of absolutely and of relatively uniform convergence of sequences of functions. For case IV we have the well known theorem:

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* As to proofs, cf. § 16.

> A uniformly convergent sequence of continuous functions has as limit a continuous function.

The corresponding proposition holds in case $\mathrm{III}_0$ but not in case $\mathrm{III}_1$. The scale function $1(\mathfrak{P})$ belongs to $\mathfrak{M}^{\mathrm{IV}}$ but neither to $\mathfrak{M}^{\mathrm{III}_0}$ nor to $\mathfrak{M}^{\mathrm{III}_1}$.

In terms of relative uniformity, however, for the three cases, we have the proposition:

> A sequence of functions of $\mathfrak{M}^{\mathrm{III}_0;\,\mathrm{III}_1;\,\mathrm{IV}}$ converging uniformly as to a function of $\mathfrak{M}^{\mathrm{III}_0;\,\mathrm{III}_1;\,\mathrm{IV}}$ has as limit a function of $\mathfrak{M}^{\mathrm{III}_0;\,\mathrm{III}_1;\,\mathrm{IV}}$.

The corresponding proposition holds in case $\mathrm{III}_e$ and evidently in cases I; $\mathrm{II}_n$; III, and in every case in which the class $\mathfrak{M}$ is the class of all functions on $\mathfrak{P}$ to $\mathfrak{A}$.

8. Similarly, we speak of the convergence of the sequence $\{\mu_n\}$

on $\mathfrak{P}$; uniformly on $\mathfrak{P}$; uniformly on $\mathfrak{P}$ as to $\sigma$:

in notation,

$$(p);\ (\mathfrak{P});\ (\mathfrak{P};\ \sigma),$$

in case there exists a function $\theta$ effective as limit of the sequence:

$$\underset{n}{\mathrm{L}}\mu_n = \theta \quad (p); \qquad \underset{n}{\mathrm{L}}\mu_n = \theta \quad (\mathfrak{P}); \qquad \underset{n}{\mathrm{L}}\mu_n = \theta \quad (\mathfrak{P};\ \sigma),$$

for the respective modes of convergence.

A condition necessary and sufficient for the convergence in the various modes is the validity in the respective modes of the so-called Cauchy condition. This condition for the relative uniformity $(\mathfrak{P};\ \sigma)$ is as follows:

$$e : \supset : \exists\ n_e \ni n' > n_e \,.\, n'' > n_e \,.\supset .\ A(\mu_{n'} - \mu_{n''}) \leqq eA\sigma,$$

that is, for every $e$ there exists an integer $n_e$ dependent on $e$ such that for every two integers $n'$, $n''$ each exceeding $n_e$ it is true that the relation: $A(\mu_{n'} - \mu_{n''}) \leqq eA\sigma$, holds.

The limit $\theta$ is of course unique. To the same function $\theta$

as limit converges in like mode every sequence of functions obtained from the original sequence by (1°) any rearrangement of terms, (2°) the omission of any finite or infinite number of terms, or (3°) the insertion of any finite number of functions as new terms.

Thus, with respect to a function $\sigma$, any class $\mathfrak{M} \equiv [\mu]$ of functions constitutes a class $(L)$ of FRÉCHET, on the understanding that a sequence $\{\mu_n\}$ has in the sense $(L)$ a limit if and only if there exists a $\mu$ (the limit) such that in our sense

$$\underset{n}{\mathrm{L}}\,\mu_n = \mu \quad (\mathfrak{P};\ \sigma).$$

9. *Definition.*—A function $\sigma$ is said to be *dominated by* a function $\tau$ in case

$$A\sigma \leqq A\tau,$$

that is, in case $A\sigma_p \leqq A\tau_p$ for every $p$.

9*a*. *Comparison of uniform and relatively uniform convergence.*—We notice the propositions:

1) Uniformity (of convergence) as to $\sigma$ implies uniformity as to every function $\tau$ dominating $\sigma$.

2) Uniformity as to $\sigma$ and uniformity as to $a\sigma$ $(a \neq 0)$ are equivalent.

3) Uniformity as to 0 implies that the functions of the seqence are ultimately all equal and hence implies uniformity as to every function $\sigma$.

4) If $A\sigma$ is uniformly bounded from $\infty$, viz.,

$$\exists\ e \ni A\sigma \leqq e,$$

then uniformity as to $\sigma$ implies uniformity. Similarly, if $A\sigma$ is uniformly bounded from 0, viz.,

$$\exists\ e \ni e \leqq A\sigma,$$

then uniformity implies uniformity as to $\sigma$.

These propositions hold conversely, viz.:

If $A\sigma$ is not uniformly bounded from $\infty$, there exists a sequence* of functions converging uniformly as to $\sigma$ but non-uniformly.

If $A\sigma$ is not uniformly bounded from 0, there exists a sequence* of functions converging uniformly but non-uniformly as to $\sigma$.

5) With respect to a scale function $\sigma$ and a positive number $e \leqq 1$ the elements $p$ of the class $\mathfrak{P}$ may be distributed into three mutually exclusive classes:†

$$\mathfrak{P}' \equiv [p'], \quad \mathfrak{P}'' \equiv [p''], \quad \mathfrak{P}''' \equiv [p'''],$$

according as

$$\frac{1}{e} < A\sigma_p, \quad e \leqq A\sigma_p \leqq \frac{1}{e}, \quad A\sigma_p < e.$$

Then uniformity (whether absolute or relative to $\sigma$) on $\mathfrak{P}$ is equivalent to the simultaneous (corresponding) uniformities on $\mathfrak{P}'$, on $\mathfrak{P}''$, and on $\mathfrak{P}'''$.

The condition of relative uniformity as to $\sigma$ is on $\mathfrak{P}''$ equivalent to, on $\mathfrak{P}'$ in general less exacting than, on $\mathfrak{P}'''$ in general more exacting than the condition of uniformity.

10. The convergence uniform as to $\sigma$ of the sequence $\{\mu_n\}$ implies for every $\tau$ the convergence uniform as to $\sigma\tau$ of the sequence $\{\mu_n\tau\}$. In particular, if $\sigma$ vanishes nowhere, the convergence uniform as to $\sigma$ of the sequence $\{\mu_n\}$ and the uniform convergence of the sequence $\{\mu_n/\sigma\}$ are equivalent. Further, the convergence of the sequence $\{a_n\}$ of numbers implies for every $\sigma$ the convergence uniform as to $\sigma$ of the sequence $\{a_n\sigma\}$ of functions.

11. With similar definitions for functions of two independent variables: $p'$ on $\mathfrak{P}'$; $p''$ on $\mathfrak{P}''$, that is, by § 4, for functions of the variable $p'p''$ on $\mathfrak{P}'\mathfrak{P}''$, we notice the propositions:

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* E. g., the respective sequences:

$$\{a_n\sigma\}; \quad \{a_n(\mathfrak{P})\},$$

where $\{a_n\}$ is a monotonic increasing sequence of numbers with limit 1.

† Of these three classes one or two may be null-classes.

1) Uniformity on $\mathfrak{P}'\mathfrak{P}''$ as to $\sigma$ implies for every $p'$ uniformity on $\mathfrak{P}''$ as to $\sigma_{p'}$, and for every $p''$ uniformity on $\mathfrak{P}'$ as to $\sigma_{p''}$.

2) Uniformity on $\mathfrak{P}'\mathfrak{P}''$ as to $\sigma'\sigma''$, where $\sigma'$ is on $\mathfrak{P}'$ and $\sigma''$ is on $\mathfrak{P}''$, implies for every $p'$ uniformity on $\mathfrak{P}''$ as to $\sigma''$, and for every $p''$ uniformity on $\mathfrak{P}'$ as to $\sigma'$.

12. With similar definitions similar propositions hold concerning the convergence uniform as to $\sigma$ of an infinite series of functions $\mu_n$:

$$\sum_n \mu_n = \theta \quad (\mathfrak{P};\ \sigma).$$

Further, the usual comparison theorem holds, viz., if the series $\sum_n \mu_n$ of functions $\mu_n$ each nowhere negative converges uniformly as to $\sigma$, then so does any series $\sum_n \mu'_n$ of functions $\mu'_n$ each dominated by the corresponding function $\mu_n$ of the series $\sum_n \mu_n$, and of the two functions $\sum_n \mu_n$, $\sum_n \mu'_n$ the former dominates the latter.

As a corollary: the convergence uniform as to $\sigma$ of $\sum_n A\mu_n$ implies like convergence of $\sum_n \mu_n$, and the function $\sum_n A\mu_n$ dominates the function $\sum_n \mu_n$. In this case, the series $\sum_n \mu_n$ is said to be absolutely-uniformly convergent on $\mathfrak{P}$ as to $\sigma$.

13. *Propositions.*—The relations:

$$\underset{n}{\mathrm{L}}\, \mu'_n = \theta' \quad (\mathfrak{P};\ \sigma'); \qquad \underset{n}{\mathrm{L}}\, \mu''_n = \theta'' \quad (\mathfrak{P};\ \sigma''),$$

imply

(1) $$\underset{n}{\mathrm{L}}\, A\mu'_n = A\theta' \quad (\mathfrak{P};\ \sigma');$$

(2) $$\underset{n}{\mathrm{L}}\, a\mu'_n = a\theta' \quad (\mathfrak{P};\ \sigma');$$

(3) $$\underset{n}{\mathrm{L}}\, (\mu'_n \pm \mu''_n) = \theta' \pm \theta'' \quad (\mathfrak{P};\ A\sigma' + A\sigma'');$$

(4) $$\underset{n}{\mathrm{L}}\, \mu'_n\mu''_n = \theta'\theta'' \quad (\mathfrak{P};\ \tau'\tau''),$$

where $\tau'$ and $\tau''$ are respectively common dominants of $\theta'$, $\sigma'$ and of $\theta''$, $\sigma''$.

More generally, from the relations:

$$\underset{n}{\mathrm{L}}\, \mu'_n = \theta' \quad (\mathfrak{P}';\ \sigma'); \qquad \underset{n}{\mathrm{L}}\, \mu''_n = \theta'' \quad (\mathfrak{P}'';\ \sigma''),$$

where the functions $\mu'_n$, $\theta'$, $\sigma'$ are on $\mathfrak{P}'$ and the functions $\mu''_n$, $\theta''$, $\sigma''$ are on $\mathfrak{P}''$, we have

$$(5) \qquad \underset{n}{\mathrm{L}}(\mu'_n \pm \mu''_n) = \theta' \pm \theta'' \ (\mathfrak{P}'\mathfrak{P}''\,;\ A\sigma' + A\sigma'')\,;$$

$$(6) \qquad \underset{n}{\mathrm{L}}\mu'_n\mu''_n = \theta'\theta'' \ (\mathfrak{P}'\mathfrak{P}''\,;\ \tau'\tau''),$$

where $\tau'$, $\tau''$ on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively are common dominants: $\tau'$ of $\theta'$, $\sigma'$ and $\tau''$ of $\theta''$, $\sigma''$.

Here the functions: $\mu'_n \pm \mu''_n$; $\theta' \pm \theta''$; $A\sigma' + A\sigma''$; $\mu'_n\mu''_n$; $\theta'\theta''$; $\tau'\tau''$, are functions on $\mathfrak{P}'\mathfrak{P}'' \equiv [p'p'']$. In case $\mathfrak{P}' = \mathfrak{P}'' = \mathfrak{P}$, the relations (3; 4) appear as corollaries of (5; 6), embracing (5; 6) for the assumption $p' = p'' = p$.

### *Classes of functions. Closure properties.* §§ 14–17.

14. *Derivative classes of functions.*—From several functions and classes of functions on $\mathfrak{P}$ to $\mathfrak{A}$ we may derive other functions and classes of functions.

With respect to

$$\mathfrak{M} \equiv [\mu], \qquad \mathfrak{M}' \equiv [\mu'], \qquad \mathfrak{M}'' \equiv [\mu''], \qquad \sigma$$

we denote by

$$A\mathfrak{M},\ \mathfrak{A}\mathfrak{M},\ \mathfrak{M}' \pm \mathfrak{M}'',\ \mathfrak{M}'\mathfrak{M}'',\ \mathfrak{M}_\sigma$$

the classes consisting of all functions of the respective forms:*

$$A\mu,\ a\mu,\ \mu' \pm \mu'',\ \mu'\mu'',\ \underset{n}{\mathrm{L}}\mu_n\ (\mathfrak{P}\,;\ \sigma).$$

Here $\mathfrak{M}_\sigma$, *the class* $\mathfrak{M}$ *extended as to the function* $\sigma$, is the class of all limit functions of sequences $\{\mu_n\}$ of the class $\mathfrak{M}$ converging on $\mathfrak{P}$ uniformly as to the function $\sigma$. Further, with respect to two classes: $\mathfrak{M} \equiv [\mu]$; $\mathfrak{S} \equiv [\sigma]$, by

$$\mathfrak{M}_{\mathfrak{S}},$$

*the class* $\mathfrak{M}$ *extended as to the class* $\mathfrak{S}$, we denote the class of all

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* In particular, we denote by

$$\mathfrak{M} \pm \mathfrak{M}, \quad \mathfrak{M}\mathfrak{M}$$

the classes consisting of all functions of the respective forms:

$$\mu_1 \pm \mu_2, \quad \mu_1\mu_2.$$

Thus $\mathfrak{M} - \mathfrak{M}$ contains the function $0(\mathfrak{P})$.

limit functions of sequences $\{\mu_n\}$ of the class $\mathfrak{M}$ converging uniformly each as to some function $\sigma$ of the class $\mathfrak{S}$. Thus the class $\mathfrak{M}_{\mathfrak{S}}$ is the least common superclass $\mathsf{U}[\mathfrak{M}_\sigma \,|\, \sigma^{\mathfrak{S}}]$ of the classes $\mathfrak{M}_\sigma$, $\sigma$ varying over the class $\mathfrak{S}$.

14*a*. *Propositions.*—Concerning these extensions as to classes or functions we have the propositions:*

(1) $$\mathfrak{M} \,.\, \sigma \,.\, \supset \,.\, \mathfrak{M}^{\mathfrak{M}_\sigma};$$

(2) $$\mathfrak{M} \,.\, \mathfrak{S} \,.\, \supset \,.\, \mathfrak{M}^{\mathfrak{M}_{\mathfrak{S}}};$$

(3) $$\mathfrak{M}'^{\,\mathfrak{M}''} .\, \mathfrak{S} \,.\, \supset \,.\, \mathfrak{M}'^{\,\mathfrak{M}''_{\mathfrak{S}}}_{\mathfrak{S}};$$

(4) $$\mathfrak{M} \,.\, \mathfrak{S}'^{\,\mathfrak{S}''} .\, \supset \,.\, \mathfrak{M}_{\mathfrak{S}'}^{\mathfrak{M}_{\mathfrak{S}''}};$$

(5) $$\mathfrak{M} \,.\, \mathfrak{S} \,.\, \supset \,.\, \mathfrak{M}_{\mathfrak{S}} = \mathfrak{M}_{\mathfrak{S}\mathfrak{S}}.$$

15. *Closure properties of classes of functions.*—A class $\mathfrak{M}$ is

| | |
|---|---|
| $A \equiv$ *absolute*, | if $\mathfrak{M}$ contains $A\mathfrak{M}$; |
| *multiplicative as to* $\mathfrak{A}$, | if $\mathfrak{M}$ contains $\mathfrak{A}\mathfrak{M}$; |
| *additive*, | if $\mathfrak{M}$ contains $\mathfrak{M} + \mathfrak{M}$; |
| $L \equiv$ *linear*, | if $\mathfrak{M}$ contains $\mathfrak{A}\mathfrak{M} + \mathfrak{A}\mathfrak{M}$; |
| *multiplicative*, | if $\mathfrak{M}$ contains $\mathfrak{M}\mathfrak{M}$; |
| $C_{\mathfrak{S}} \equiv$ *closed as to* $\mathfrak{S}$, | if $\mathfrak{M}$ contains $\mathfrak{M}_{\mathfrak{S}}$; |
| $C \equiv$ *closed*,† | if $\mathfrak{M}$ contains $\mathfrak{M}_{\mathfrak{M}}$. |

In accordance with § 2, the notations:

$$\mathfrak{M}^A; \ \mathfrak{M}^{\text{additive}}; \ \mathfrak{M}^L; \ \text{etc.},$$

designate classes $\mathfrak{M}$ respectively absolute; additive; linear; etc.

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[*] 1) For every class $\mathfrak{M}$ and function $\sigma$ the class $\mathfrak{M}$ belongs to the class $\mathfrak{M}_\sigma$.

2) For every class $\mathfrak{M}$ and class $\mathfrak{S}$ the class $\mathfrak{M}$ belongs to the class $\mathfrak{M}_{\mathfrak{S}}$.

3) For every class $\mathfrak{S}$ and class $\mathfrak{M}'$ belonging to a class $\mathfrak{M}''$ the class $\mathfrak{M}'_{\mathfrak{S}}$ belongs to the class $\mathfrak{M}''_{\mathfrak{S}}$.

4) For every class $\mathfrak{M}$ and class $\mathfrak{S}'$ belonging to a class $\mathfrak{S}''$ the class $\mathfrak{M}_{\mathfrak{S}'}$ belongs to the class $\mathfrak{M}_{\mathfrak{S}''}$.

5) For every class $\mathfrak{M}$ and class $\mathfrak{S}$ the classes $\mathfrak{M}_{\mathfrak{S}}$ and $\mathfrak{M}_{\mathfrak{S}\mathfrak{S}}$ are identical.

† A class $\mathfrak{M}$ having the property $C_{\mathfrak{M}}$ of being closed as to itself has the property $C$, and conversely. The notation $C$ and the designation *closure* are preferable to the *S-C* and *self-closure* originally used.

A class $\mathfrak{M}$ on $\mathfrak{P}^{\mathrm{II}_n}$ is closed if and only if the corresponding $n$-dimensional point-set is closed, i. e., contains its derived point-set, according to the usual definitions.

The linear classes are the classes additive and multiplicative as to constants. A linear multiplicative class is *integral.*

Since for every $\mathfrak{S}$, $\mathfrak{M}_{\mathfrak{S}}$ contains $\mathfrak{M}$, for a class $\mathfrak{M}$ the properties: closure; closure as to $\mathfrak{S}$, are respectively equivalent to invariance under extension as to $\mathfrak{M}$; $\mathfrak{S}$, viz., $\mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}$; $\mathfrak{M}'_{\mathfrak{S}} = \mathfrak{M}$.

16. *The classes* $\mathfrak{M}^{\mathrm{I};\, \mathrm{II}_n;\, \mathrm{III};\, \mathrm{III}_0;\, \mathrm{III}_c;\, \mathrm{IV}}$ *are absolute, linear and closed.** — They are moreover multiplicative and so integral, but these facts do not enter the theory of the present paper.

A class $\mathfrak{M}$ closed as to $\mathfrak{A}(\mathfrak{P})$ is a class $\mathfrak{M}$ containing all functions expressible as limits of uniformly converging sequences of functions of $\mathfrak{M}$, and conversely.

The classes $\mathfrak{M}^{\mathrm{I}\text{-}\mathrm{IV}}$ with the exception of $\mathfrak{M}^{\mathrm{III}_0}$, $\mathfrak{M}^{\mathrm{III}_c}$ contain $\mathfrak{A}(\mathfrak{P})$ and being closed they are closed as to $\mathfrak{A}(\mathfrak{P})$.

The class $\mathfrak{M}^{\mathrm{III}_0}$ does not contain but is closed as to $\mathfrak{A}(\mathfrak{P})$.

The class $\mathfrak{M}^{\mathrm{III}_e}$ neither contains nor is closed as to $\mathfrak{A}(\mathfrak{P})$. For consider the function $\theta$ and the sequence $\{\mu_n\}$ of functions $\mu_n$ defined as follows:

$$\theta_p \equiv p^{-1/e} \quad (p);$$

$$\mu_{np} \equiv p^{-1/e} \quad (p \leqq n), \qquad 0 \quad (p > n).$$

Then obviously the functions $\mu_n$ do and the function $\theta$ does not belong to the class $\mathfrak{M}^{\mathrm{III}_e}$, while the sequence $\{\mu_n\}$ converges on $\mathfrak{P}$ uniformly to the function $\theta$.

In the light of this remark as to $\mathfrak{M}^{\mathrm{III}_e}$ one may appreciate the importance for General Analysis of the generalization of uniformity to relative uniformity.

17. The properties of classes $\mathfrak{M}$ defined in § 15 are *closure properties,* in the sense that a class $\mathfrak{M}$ having the property contains a certain class dependent on $\mathfrak{M}$ and the property in question.

Thus, the (specific) *closure* $C$ of a class $\mathfrak{M}$ is the property:

$$\mathop{\mathrm{L}}_{n} \mu_n = \theta \quad (\mathfrak{P};\, \mu) \,.\supset.\, \theta^{\mathfrak{M}},$$

viz., *a sequence* $\{\mu_n\}$ *of functions of* $\mathfrak{M}$ *converging on* $\mathfrak{P}$ *uniformly as to a function* $\mu$ *of* $\mathfrak{M}$ *converges to a limit* $\theta$ *which belongs to* $\mathfrak{M}$.

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* Cf. the theorem of § 23.

That these closure properties enter fundamentally in the theory of functions of the classes $\mathfrak{M}^{\mathrm{I-IV}}$ is evident. An approach to the theory of convergent sequences of continuous functional transformations or to the theory of other forms of double limits reveals the presence of another fundamental property held in common by these classes, a *dominance property D.*

### *Dominance properties. Relative uniformity as to a class of functions.* §§ 18–26.

18. *A function $\sigma$ is dominated by a function $\tau$* in case $A\sigma \leqq A\tau$ (cf. § 9), and *by a class* $\mathfrak{T} = [\tau]$ in case in the class $\mathfrak{T}$ there is a function $\tau_\sigma$ (dependent upon $\sigma$) by which $\sigma$ is dominated.
*A class* $\mathfrak{S} \equiv [\sigma]$ *is dominated by a function* $\tau$ or *a class* $\mathfrak{T}$ in case its every function $\sigma$ is dominated by $\tau$ or by $\mathfrak{T}$.

We use temporarily the notations:

$$\sigma^{\text{dominated by } \tau}; \qquad \sigma^{\text{dominated by } \mathfrak{T}};$$

$$\mathfrak{S}^{\text{dominated by } \tau}; \qquad \mathfrak{S}^{\text{dominated by } \mathfrak{T}}.$$

19. *Dominance property D.*—A class $\mathfrak{M}$ has the *dominance property D* in case for every sequence $\{\mu_n\}$ of functions $\mu_n \equiv (\mu_{np} \mid p)$ of $\mathfrak{M}$ there exists a function $\mu \equiv (\mu_p \mid p)$ of $\mathfrak{M}$ such that every function $\mu_n$ of the sequence is dominated by the class $\mathfrak{A}\mu$, that is, such that there exists a sequence $\{a_n\}$ of real numbers of such a nature that for every $n$ and $p$

$$A\mu_{np} \leqq Aa_n\mu_p.$$

In symbols:

$$\mathfrak{M}^D : \equiv : \mathfrak{M} \ni \{\mu_n\} \,.\supset.\, \exists\ (\{a_n\}, \mu) \ni A\mu_{np} \leqq Aa_n\mu_p \quad (np).$$

20. *Relative uniformity as to a class of functions.*—An analytic relation involving a parameter $p$ is said to *hold uniformly on a range $\mathfrak{P}$ of $p$ with respect to a class of functions on $\mathfrak{P}$ as a scale of uniformity* in case it holds uniformly as to some function of the class.

Thus, if $\mathfrak{S} \equiv [\sigma]$,

$$\mathbf{L}_n \mu_n = \theta \quad (\mathfrak{P};\ \mathfrak{S})$$

and

$$\exists\ \sigma \ni \mathop{\mathbf{L}}_{n} \mu_n = \theta \quad (\mathfrak{P};\ \sigma)$$

are equivalent statements.

20*a*. *Propositions of comparison.*

1) Uniformity as to $\mathfrak{S}$ and uniformity as to $\mathfrak{A}\mathfrak{S}$ are equivalent.

2) If and only if* $\mathfrak{A}\mathfrak{S}$ is dominated by $\mathfrak{A}\mathfrak{T}$ does uniformity as to $\mathfrak{S}$ imply uniformity as to $\mathfrak{T}$.

3) Uniformity and uniformity as to $\mathfrak{A}(\mathfrak{P})$ are equivalent. Here, according to § 6, $\mathfrak{A}(\mathfrak{P})$ denotes the class of constant functions of the argument $p$.

4) If and only if $\mathfrak{A}(\mathfrak{P})$ is dominated by $\mathfrak{A}\mathfrak{S}$ does uniformity imply uniformity as to $\mathfrak{S}$.

5) If and only if $\mathfrak{A}\mathfrak{S}$ is dominated by $\mathfrak{A}(\mathfrak{P})$ does uniformity as to $\mathfrak{S}$ imply uniformity.

6) The classes $\mathfrak{M}=\mathfrak{M}^{\mathrm{I};\ \mathrm{II}_n;\ \mathrm{IV}}$ are dominated by and dominate $\mathfrak{A}(\mathfrak{P})$, and accordingly uniformity as to $\mathfrak{M}$ and uniformity are equivalent.

The class $\mathfrak{M}=\mathfrak{M}^{\mathrm{III}}$ dominates but is not dominated by $\mathfrak{A}(\mathfrak{P})$, and accordingly uniformity implies but is not implied by uniformity as to $\mathfrak{M}$.

The classes $\mathfrak{M}=\mathfrak{M}^{\mathrm{III}_0;\ \mathrm{III}_e}$ are dominated by but do not dominate $\mathfrak{A}(\mathfrak{P})$, and accordingly uniformity as to $\mathfrak{M}$ implies but is not implied by uniformity.

7) The convergence uniform as to $\mathfrak{S}$ of the sequence $\{\mu_n\}$ and the corresponding validity of the Cauchy condition are equivalent.

21. *The dominance property D enters the theory chiefly as a property of classes functioning as scales of uniformity.* Before

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* The condition is evidently sufficient. In proof of the necessity, note that

$$\mathop{\mathbf{L}}_{n} \frac{1}{n} = 0 \quad (\mathfrak{P}), \qquad \sigma . \supset . \mathop{\mathbf{L}}_{n} \frac{\sigma}{n} = 0 \quad (\mathfrak{P};\sigma),$$

and further that

$$\mathop{\mathbf{L}}_{n} \frac{\sigma}{n} = 0 \quad (\mathfrak{P};\mathfrak{T}) \quad \text{implies} \quad \exists\ \tau \ni \mathop{\mathbf{L}}_{n} \frac{\sigma}{n} = 0 \quad (\mathfrak{P};\tau),$$

so that, for $n$ sufficiently large, $\varDelta\sigma \leqq n\varDelta\tau$; and accordingly, if uniformity as to $\mathfrak{S}$ implies uniformity as to $\mathfrak{T}$, every function $\sigma$ of $\mathfrak{S}$ is dominated by $\mathfrak{A}\mathfrak{T}$.

indicating propositions in this direction I define other dominance properties of classes of functions.

22. *Dominance properties* $D_0$, $D'_0$, $D_1$, $D_2$.—A class $\mathfrak{M} \equiv [\mu]$ has the *dominance property*

$D_0$: in case for every function $\mu$ of $\mathfrak{M}$ there is a nowhere negative function $\mu_0$ of $\mathfrak{M}$ which dominates $\mu$;

$D'_0$: in case $\mathfrak{M}$ contains every function $\theta$ such that there exists a nowhere negative function $\mu_0$ of $\mathfrak{M}$ dominating $\theta$;

$D_1$: in case every two functions of $\mathfrak{M}$ have in $\mathfrak{A}\mathfrak{M}$ a common dominant function;

$D_2$: in case there exists a denumerable subclass $\mathfrak{N} \equiv [\nu_n | n]$ of the class $\mathfrak{A}\mathfrak{M}$ such that $\mathfrak{M}$ is dominated by $\mathfrak{N}$.

22*a*. *Propositions* concerning the dominance properties* $D$, $D_0$, $D'_0$, $D_1$, $D_2$.—

(1) $$\mathfrak{M}^{P} .. \sim . (\mathfrak{A}\mathfrak{M})^{P} \quad (P = D,\ D_1,\ D_2).$$

(2) $$\mathfrak{M}_0^{\mathfrak{M} \cdot P} . \mathfrak{M}^{\text{dominated by } \mathfrak{M}_0} . \supset . \mathfrak{M}^{P} \quad (P = D,\ D_0,\ D_1,\ D_2).$$

(2′) $$\mathfrak{M}_0^{\mathfrak{M} \cdot P} . \mathfrak{M}^{\text{dominated by } \mathfrak{A}\mathfrak{M}_0} . \supset . \mathfrak{M}^{P} \quad (P = D,\ D_1,\ D_2).$$

(3) $$\mathfrak{M}^{\mathfrak{N}} . \supset . \mathfrak{M}^{\text{dominated by } \mathfrak{N}}.$$

(4) $$\mathfrak{M} . \mathfrak{S}_0^{\mathfrak{S}} . \mathfrak{S}^{\text{dominated by } \mathfrak{A}\mathfrak{S}_0} . \supset . \mathfrak{M}_{\mathfrak{S}_0} = \mathfrak{M}_{\mathfrak{S}}.$$

(5) $$\mathfrak{M}^{A} . \supset . \mathfrak{M}^{D_0}.$$

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* 1) For the respective dominance properties $P = D$; $D_1$; $D_2$, a class $\mathfrak{M}$ has the property $P$ if and only if the class $\mathfrak{A}\mathfrak{M}$ has the property $P$.

2) For the respective dominance properties $P = D$; $D_0$; $D_1$; $D_2$, if a class $\mathfrak{M}$ is dominated by one of its subclasses $\mathfrak{M}_0$ having the property $P$, then $\mathfrak{M}$ itself has the property $P$.

2′) For the respective dominance properties $P = D$; $D_1$; $D_2$, if a class $\mathfrak{M}$ is dominated by the class $\mathfrak{A}\mathfrak{M}_0$, where $\mathfrak{M}_0$ is a subclass of $\mathfrak{M}$ having the property $P$, then $\mathfrak{M}$ has the property $P$.

3) If $\mathfrak{M}$ is a subclass of $\mathfrak{N}$, then $\mathfrak{M}$ is dominated by $\mathfrak{N}$.

4) If $\mathfrak{S}_0$ belongs to and $\mathfrak{A}\mathfrak{S}_0$ dominates $\mathfrak{S}$, then for every class $\mathfrak{M}$ the extensions $\mathfrak{M}_{\mathfrak{S}_0}$ ; $\mathfrak{M}_{\mathfrak{S}}$ of $\mathfrak{M}$ as to $\mathfrak{S}_0$ and $\mathfrak{S}$ are identical.

5) Every absolute class has the dominance property $D_0$.

6) Every absolute linear class has the dominance properties $D_0$, $D_1$.

7) If a class $\mathfrak{M}$ is closed and a class $\mathfrak{N}$ has the dominance property $D_1$, then the subclass of $\mathfrak{M}$ consisting of all its functions $\mu$ which are dominated by $\mathfrak{A}\mathfrak{N}$ is closed.

7′) If a class $\mathfrak{N}$ has the dominance property $D_1$, then the class consisting of all functions dominated by $\mathfrak{A}\mathfrak{N}$ is closed.

8) Every absolute linear class with the dominance property $D'_0$ is closed.

(6) $$\mathfrak{M}^{AL} . \supset . \mathfrak{M}^{D_0 D_1}.$$

(7) $$\mathfrak{M}^{C} . \mathfrak{N}^{D_1} . \supset . [\text{all } \mu^{\text{dominated by } \mathfrak{A}\mathfrak{M}}]^{C}.$$

In particular, $\mathfrak{M}$ being the class of all functions $\phi$ on $\mathfrak{P}$ to $\mathfrak{A}$,

(7′) $$\mathfrak{N}^{D_1} . \supset . [\text{all } \phi^{\text{dominated by } \mathfrak{A}\mathfrak{M}}]^{C}.$$

(8) $$\mathfrak{M}^{ALD_0'} . \supset . \mathfrak{M}^{C}.$$

Here (8) is a corollary of (6, 7′), since

$$\mathfrak{M} = \mathfrak{A}\mathfrak{M} = A\mathfrak{M} = [\text{all } \phi^{\text{dominated by } \mathfrak{M}}].$$

23. The class $\mathfrak{M}^{\mathrm{I}} \equiv \mathfrak{A}$ and similarly every class $\mathfrak{A}(\mathfrak{P})$ has the properties $D$, $D_0$, $D_1$, $D_2$. Accordingly, by § 22*a*2, *these properties belong* to every class $\mathfrak{M}$ dominated by and containing $\mathfrak{A}(\mathfrak{P})$, in particular *to the classes* $\mathfrak{M} = \mathfrak{M}^{\mathrm{I};\ \mathrm{II}_n;\ \mathrm{IV}}$.

Further, *the classes* $\mathfrak{M} = \mathfrak{M}^{\mathrm{III};\ \mathrm{III}_0;\ \mathrm{III}_e}$ *have the properties* $A$, $L$, $D_0$, $D_0'$, $D_1$.

We formulate the

Theorem. *The properties* $A$, $L$, $C$, $D$, $D_0$, $D_1$ *are possessed by the classes* $\mathfrak{M} = \mathfrak{M}^{\mathrm{I};\ \mathrm{II}_n;\ \mathrm{III};\ \mathrm{III}_0;\ \mathrm{III}_e;\ \mathrm{IV}}$.

*The property* $D_2$ *is possessed by the classes* $\mathfrak{M} = \mathfrak{M}^{\mathrm{I};\ \mathrm{II}_n;\ \mathrm{IV}}$, *but not by the classes* $\mathfrak{M} = \mathfrak{M}^{\mathrm{III};\ \mathrm{III}_0;\ \mathrm{III}_e}$.

*The property* $D_0'$ *is possessed by the classes* $\mathfrak{M} = \mathfrak{M}^{\mathrm{I};\ \mathrm{II}_n;\ \mathrm{III};\ \mathrm{III}_0;\ \mathrm{III}_e}$, *but not by the class* $\mathfrak{M}^{\mathrm{IV}}$.

*The composite property* $D_0' D_2$ *is possessed by the classes* $\mathfrak{M} = \mathfrak{M}^{\mathrm{I};\ \mathrm{II}_n}$, *but not by the classes* $\mathfrak{M} = \mathfrak{M}^{\mathrm{III};\ \mathrm{III}_0;\ \mathrm{III}_e;\ \mathrm{IV}}$.

23*a*. The two dominance properties $D_0'$, $D_2$ determine our composite properties $P$:

$$P = D_0' D_2 ;\ D_0'^{-} D_2 ;\ {}^{-}D_0' D_2 ;\ {}^{-}D_0'^{-} D_2,$$

and corresponding genera of classes $\mathfrak{M}$ or systems $(\mathfrak{A} ; \mathfrak{P} ; \mathfrak{M})$, the genus $P$ consisting of all classes $\mathfrak{M}$ having the property $P$.

Thus the properties $D_0'$, $D_2$ serve as *branching properties*[*]

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[*] These branching properties are properties of general reference. Another branching property of general reference is $C_{\mathfrak{A}(\mathfrak{P})}$; this property is possessed by the classes $\mathfrak{M} = \mathfrak{M}^{\mathrm{I};\ \mathrm{II}_n;\ \mathrm{III};\ \mathrm{III}_0;\ \mathrm{IV}}$ but not by the class $\mathfrak{M}^{\mathrm{III}_e}$ (cf. §§ 7f, 16) Other branchings relate to forms of expressibility (e. g., linear expressibility) of the functions of the class $\mathfrak{M}$ in terms of a system of fundamental functions belonging to the class $\mathfrak{M}$ itself or to an auxiliary class $\mathfrak{S}$.

for the classes $\mathfrak{M} = \mathfrak{M}^{\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III};\,\mathrm{III}_0;\,\mathrm{III}_e;\,\mathrm{IV}}$, viz.,

*Classes* $\mathfrak{M}$: $\quad \mathfrak{M}^{\mathrm{I};\,\mathrm{II}_n}$; $\mathfrak{M}^{\mathrm{III};\,\mathrm{III}_0;\,\mathrm{III}_e}$; $\mathfrak{M}^{\mathrm{IV}}$,

*Genus* $P$: $\quad D_0' D_2$; $D_0' \ \ D_2$; $^-D_0' D_2$.

23*b*. The class $\mathfrak{M}$ of all functions on a class $\mathfrak{P}$ is obviously closed. Thus the classes $\mathfrak{M} = \mathfrak{M}^{\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III}}$ are closed. The closure of $\mathfrak{M}^{\mathrm{IV}}$ depends upon two fundamental properties of the class $\mathfrak{M}^{\mathrm{IV}}$ of all continuous functions $\mu$ on the finite interval $\mathfrak{P}^{\mathrm{IV}}$, viz., (1) $\mathfrak{M}^{\mathrm{IV}}$ contains and is dominated by $\mathfrak{A}(\mathfrak{P})$; (2) a sequence $\{\mu_n\}$ of functions $\mu_n$ of $\mathfrak{M}^{\mathrm{IV}}$ converging uniformly on $\mathfrak{P}^{\mathrm{IV}}$ converges to a function of $\mathfrak{M}^{\mathrm{IV}}$. The closure of $\mathfrak{M}^{\mathrm{III}_0}$ and $\mathfrak{M}^{\mathrm{III}_e}$ depends upon their properties $ALD_0'$ in view of § 22*a*8.

That the classes $\mathfrak{M} = \mathfrak{M}^{\mathrm{III};\,\mathrm{III}_0;\,\mathrm{III}_e}$ have the property $D$ but not the property $D_2$ is proved in § 23*c*.

23*c*. *Dominance properties* $K_1$, $K_0$ *of classes* $\mathfrak{M}$ *on* $\mathfrak{P}^{\mathrm{III}}$. — For classes $\mathfrak{M}$ of functions on $\mathfrak{P} \equiv \mathfrak{P}^{\mathrm{III}}$ we define two dominance properties* $K_1$, $K_0$ having relations to the properties $D$, $D_2$.

A class $\mathfrak{M} \equiv [\mu]$ has the *dominance property*

$K_1$: in case for every function $\mu$ of $\mathfrak{M}$ there is a function $\mu_0$ of $\mathfrak{M}$ such that for every $e$ there is an element $p_e$ (dependent on $e$) such that $p \geqq p_e$ implies $A\mu_p \leqq eA\mu_{0p}$; viz.,

$$\mu \therefore \supset \therefore \exists\ \mu_0 \ni e : \supset : \exists\ p_e \ni p \geqq p_e . \supset . A\mu_p \leqq eA\mu_{0p} ;$$

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* The properties: $K_1$; $K_0$, quâ properties of classes $\mathfrak{M}$ of functions on the special class $\mathfrak{P}^{\mathrm{III}}$, are of special reference. The property $K_1$ is later developed (cf. §§ 65, 67, 72, 75, 77) into a property $K_1$ of more general reference. The two properties: $K_1$; $K_2$, of § 77 are defined for classes $\mathfrak{M}$ of functions on any class $\mathfrak{P}$ with respect to a development $\Delta$ of $\mathfrak{P}$. The notion *development* (§ 75) is a generalization of the denumeration of $\mathfrak{P}^{\mathrm{III}}$ and likewise a generalization of the partition by sequential halving of $\mathfrak{P}^{\mathrm{IV}}$. The property $K_2$ is a generalization of the property of uniform continuity on $\mathfrak{P}^{\mathrm{IV}}$. The properties: $K_1$; $K_2$, play a central rôle in the theory of composition of classes of functions of independent variables (cf. Part II).

$K_0$: in case for every sequence $\{\mu_n\}$ of functions of $\mathfrak{M}$ there is a function $\mu_0$ of $\mathfrak{M}$ such that for every $n$ and $e$ there is an element $p_{ne}$ (dependent on $n$ and $e$) such that $p > p_{ne}$ implies $A\mu_{np} \leqq eA\mu_{0p}$; viz.,

$$\{\mu_n\} \therefore \supset \therefore \exists\, \mu_0 \ni n \,.\, e : \supset : \exists\, p_{ne} \ni p > p_{ne} \,.\supset .\, A\mu_{np} \leqq eA\mu_{0p}.$$

We have for $\mathfrak{P} \equiv \mathfrak{P}^{\text{III}}$ the *propositions*:[*]

(1) $$\mathfrak{M}^{K_0} . \supset . \mathfrak{M}^{K_1};$$

(2) $$\mathfrak{M}^{K_0 D_2} : \supset : \mu \,.\supset .\, \mu^{\text{ultimately zero}};$$

(3) $$\mathfrak{M}^{K_1 D} . \supset . \mathfrak{M}^{K_0};$$

(4) $$\mathfrak{M}^{ALK_0} \ni \exists\, \mu^{\text{nowhere zero}} . \supset . \mathfrak{M}^{D};$$

(5) $$\mathfrak{M} = \mathfrak{M}^{\text{III};\,\text{III}_0;\,\text{III}_e} . \supset . \mathfrak{M}^{K_1 K_0 D - D_2}.$$

Proof of (1): $\mathfrak{M}^{K_0}$ *implies* $\mathfrak{M}^{K_1}$.—The property $K_1$ is evidently a special case of the property $K_0$.

Proof of (2).—The class $\mathfrak{M}$, having the property $D_2$, is dominated by a sequence $\{\nu_n\}$ of functions of the class $\mathfrak{A}\mathfrak{M}$. The class $\mathfrak{M}$ having also the property $K_0$, we see that for the sequence $\{\nu_n\}$ there exists a function $\mu_0$ of $\mathfrak{M}$ and for every $n$ and $e$ an integer $p_{ne}$ such that

$$n \,.\, e \,.\, p > p_{ne} \,.\supset .\, A\nu_{np} \leqq eA\mu_{0p}.$$

Now the function $\mu_0$ is dominated by a function, say $\nu_m$, of the sequence $\{\nu_n\}$, viz.,

$$p \,.\supset\, A\mu_{0p} \leqq A\nu_{mp}.$$

Accordingly

$$e \,.\, p > p_{me} \,.\supset .\, A\mu_{0p} \leqq eA\mu_{0p}.$$

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[*] 1) Every class $\mathfrak{M}$ having the property $K_0$ has the property $K_1$

2) Every class $\mathfrak{M}$ having the properties $K_0$, $D_2$ consists of functions $\mu$ ultimately zero, viz., for every function $\mu$ there exists an integer $n$ such that $\mu$ vanishes for every argument $p > n$. We notice further that the class consisting of all such ultimately vanishing functions has the properties $K_0$, $D_2$.

3) Every class $\mathfrak{M}$ having the properties $K_1$, $D$ has the property $K_0$.

4) Every absolute linear class $\mathfrak{M}$ having the property $K_0$ and containing a function $\mu$ nowhere zero has the property $D$.

5) The classes $\mathfrak{M} = \mathfrak{M}^{\text{III};\,\text{III}_0;\,\text{III}_e}$ have the properties: $K_1$; $K_0$; $D$, but not the property $D_2$.

Hence, the function $\mu_0$ is ultimately zero. Accordingly, the functions $\nu_n$ are ultimately zero, and the functions $\mu$, each dominated by a function $\nu_n$, are ultimately zero.

Proof of (3): $\mathfrak{M}^{K_1 D}$ *implies* $\mathfrak{M}^{K_0}$.—Consider a sequence $\{\mu_n\}$ of the class $\mathfrak{M}^{K_1 D}$. By the property $D$ there is for this sequence a certain function $\mu$ and by $K_1$ there is for this function $\mu$ a certain function $\mu_0$. This function $\mu_0$ is effective as the function $\mu_0$ of the property $K_0$ for the sequence $\{\mu_n\}$.

Proof of (4).—Consider a sequence $\{\mu_n\}$ of the class $\mathfrak{M}$. By the property $K_0$ there is for this sequence a certain function $\mu_0$, and by the hypothesis there is in $\mathfrak{M}$ a nowhere vanishing function $\mu$. The function $A\mu_0 + A\mu$ belongs to the class $\mathfrak{M}$, which is absolute and linear, and this function is effective as the function $\mu$ of the property $D$ for the sequence $\{\mu_n\}$.

Proof of (5).—Since the classes $\mathfrak{M} = \mathfrak{M}^{III;\, III_0;\, III_e}$ are absolute and linear and contain functions nowhere zero, in view of (1, 2, 3, 4), (5) follows from

(6) $$\mathfrak{M}^{III} . \supset . \mathfrak{M}^{K_0} ;$$

(7) $$\mathfrak{M}^{III_0} . \supset . \mathfrak{M}^{K_0} ;$$

(8) $$\mathfrak{M}^{III_e} . \supset . \mathfrak{M}^{K_1 D} .$$

Proof of (6): $\mathfrak{M}^{III}$ *implies* $\mathfrak{M}^{K_0}$.—Consider a sequence $\{\mu_n\}$ of $\mathfrak{M}^{III}$. For it there exists a function $\theta$ on $\mathfrak{P}^{III}$ such that

$$p : \supset : n \leqq p . \supset . A\theta_p \geqq pA\mu_{np} .$$

$\mathfrak{M}^{III}$ being the class of all functions on $\mathfrak{P}^{III}$, such a function $\theta$ belongs to $\mathfrak{M}^{III}$; it is effective as the function $\mu_0$ of property $K_0$ for the sequence $\{\mu_n\}$, $p_{ne}$ being any integer greater than $n$ and $1/e$.

Proof of (7): $\mathfrak{M}^{III_0}$ *implies* $\mathfrak{M}^{K_0}$.—Consider a sequence $\{\mu_n\}$ of $\mathfrak{M}^{III_0}$, that is, such that

$$\mathbf{L}_p A\mu_{np} = 0 \quad (n).$$

Hence there exists a double sequence $\{p_{nm}\}$ such that

$$n \,.\, m \,.\, p \geqq p_{nm} \,.\, \supset .\, A\mu_{np} \leqq m^{-2},$$

and so a sequence $\{p_m \,|\, m = 0, 1, 2, \cdots\}$ for which

$$p_0 = 1\,; \qquad n \leqq m \,.\, \supset .\, p_m \geqq p_{nm}\,; \qquad m > 0 \,.\, \supset .\, p_m > p_{m-1}.$$

Hence it follows that

$$n \leqq m \,.\, p \geqq p_m \,.\, \supset .\, A\mu_{np} \leqq m^{-2}.$$

Now the function $\theta$:

$$\theta_p = 1 \quad (p_0 = 1 \leqq p < p_1)\,; \qquad \theta_p = m^{-1} \quad (m > 0\,;\ p_m \leqq p < p_{m+1}),$$

connected with such a sequence $\{p_m\}$ has

$$\mathop{\mathbf{L}}_{p=\infty} \theta_p = 0,$$

and belongs to $\mathfrak{M}^{\mathrm{III}_0}$.

This function $\theta$ is effective as the function $\mu_0$ of the property $K_0$ for the sequence $\{\mu_n\}$. For we have the relation:

$$n \leqq m \,.\, p_{m+1} > p \geqq p_m \,.\, \supset .\, A\mu_{np} \leqq m^{-1} A\theta_p\,.$$

Take two integers $n$, $m_0$ with $n \leqq m_0$. From the preceding relation for the various integers $m \geqq m_0$ and the corresponding ranges $p_{m+1} > p \geqq p_m$ of $p$ we see that

$$n \leqq m_0 \,.\, p \geqq p_{m_0} \,.\, \supset .\, A\mu_{np} \leqq m_0^{-1} A\theta_p\,.$$

Accordingly, for given $n$ and $e$, if we take $m_0$ as the least integer exceeding the numbers: $n$; $e^{-1}$, and then define $p_{ne}$ as $p_{m_0}$, we have the desired relation:

$$p \geqq p_{ne} \,.\, \supset .\, A\mu_{np} \leqq eA\theta_p\,.$$

Proof of $(8_1)$: $\mathfrak{M}^{\mathrm{III}_e}$ *implies* $\mathfrak{M}^{K_1}$.—$\mathfrak{M}^{\mathrm{III}_e}$ is the class of all functions $\mu$ on $\mathfrak{P}^{\mathrm{III}}$ for which

$$\sum_p A\mu_p^{\,e} \qquad\qquad \Big(\sum_p \equiv \sum_{p=1}^{\infty}\Big)$$

converges. Since $\mathfrak{M}^{\mathrm{III}_e}$ is absolute and linear and contains nowhere vanishing functions, the proposition for the class $\mathfrak{M}^{\mathrm{III}_e}$ is a corollary of the proposition for the subclass consisting of all everywhere positive functions of $\mathfrak{M}^{\mathrm{III}_e}$. Further, the proposition for the general case $e = e$ is a corollary of the proposition for the case $e = 1$. We are then to prove the proposition for the class of convergent series of positive terms.

Let $\mu$ be any everywhere positive function with convergent series:

$$\sum_p \mu_p .$$

Setting

$$\rho_p = \mu_p + \mu_{p+1} + \cdots \quad (p)$$

we have

$$\mathop{\mathbf{L}}_p \rho_p = 0, \qquad \mu_p = \rho_p - \rho_{p+1} .$$

The function $\theta$:

$$\theta_p = \sqrt{\rho_p} - \sqrt{\rho_{p+1}} \quad (p),$$

is everywhere positive with convergent series:

$$\sum_p \theta_p = \sqrt{\rho_1},$$

while

$$\mathop{\mathbf{L}}_p \frac{\mu_p}{\theta_p} = \mathop{\mathbf{L}}_p (\sqrt{\rho_p} + \sqrt{\rho_{p+1}}) = 0.$$

Thus, the function $\theta$ is effective as the function $\mu_0$ of the property $K_1$ for the function $\mu$ of the class of convergent series of positive terms.

Proof of $(8_2)$: $\mathfrak{M}^{III}$ *implies* $\mathfrak{M}^{D}$. — Making the same reductions as in the proof of $(8_1)$, we consider any sequence $\{\mu_n\}$ of everywhere positive functions with convergent series. We set

$$\sum_p \mu_{np} = s_n \quad (n).$$

Let $\{b_n\}$ be a sequence of positive numbers with convergent series:

$$\sum_n b_n = b.$$

Consider the double sequence:

$$\{c_{np}\} : \quad c_{np} = b_n \frac{\mu_{np}}{s_n} \quad (np).$$

We have

$$\sum_p c_{np} = b_n \quad (n); \qquad \sum_n \sum_p c_{np} = \sum_n b_n = b,$$

and accordingly

$$\sum_p \sum_n c_{np} = b.$$

Thus there is a function $\theta$:

$$\theta_p = \sum_n c_{np} \quad (p),$$

everywhere positive with convergent series:

$$\sum_p \theta_p = b.$$

This function $\theta$ is effective as the function $\mu$ of the property $D$ for the sequence $\{\mu_n\}$. For, setting

$$a_n = \frac{s_n}{b_n} \quad (n),$$

we have the desired relation:

$$A\mu_{np} \leqq Aa_n\theta_p \quad (np),$$

since $\mu_{np} = a_n c_{np}$ $(np)$ and $c_{np} < \theta_p$ $(np)$.

23*d*. *Remarks on the properties:* $K_0$; $K_1$, *and the propositions of* § 23*c*.

The properties: $D$; $D_2$, are of general reference while the properties: $K_0$; $K_1$, are of special reference, in that they refer to classes $\mathfrak{M}$ on $\mathfrak{P}^{\mathrm{III}}$. I introduce the properties $K_0$, $K_1$ in order to make the convenient arrangement (§ 23*c*) of the proofs of the propositions of § 23 concerning the properties: $D$; $D_2$, for $\mathfrak{P} \equiv \mathfrak{P}^{\mathrm{III}}$. Later, in the theory of functions of two variables, the property $K_1$ recurs (§ 65*b*) and is developed into a property $K_1$ of more general reference (§ 77).

So far as I know the properties $D$, $D_2$ are new.

The theory of orders of infinites and infinitesimals contains the propositions that the class $\mathfrak{M}^{\mathrm{III}}$: of all numerical sequences, and the class $\mathfrak{M}^{\mathrm{III}_0}$: of all numerical sequences with limit 0, have the properties $K_1$ and $K_0$;[*] cf. proposition (5).

The proposition that the class $\mathfrak{M}^{\mathrm{III}_1}$ of all absolutely convergent series has the property $K_1$ is in effect proposition $(8_1)$

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[*] As to the property $K_0$ of the class $\mathfrak{M}^{\mathrm{III}_0}$, cf. DU BOIS-REYMOND, Mathematische Annalen, vol. 8 (1875), p. 365.

and the theorem of du Bois-Reymond:* *There is no last convergent series of positive terms.* This phrasing of the theorem and the proof given above seem to be due to Thomae,† the proof occurring however independently to Hadamard.‡

Of this proposition of du Bois-Reymond the proposition that the class $\mathfrak{M}^{III_1}$ has the property $K_0$ is a direct generalization and likewise a corollary, in view of (3, $8_2$). Hadamard§ proves the general proposition directly; he considers however only those sequences $\{\mu_n\}$ which satisfy the condition that for every $p$ $\mu_{np}$ increases with $n$.

24. *Propositions.*—

(1) $$\mathfrak{M}^{D} . \supset . \mathfrak{M}^{D_1}.$$

(2) $$\mathfrak{M}^{AD} . \supset . \mathfrak{M}^{D_0 D_1}.$$

Further, by § 13, the relations:

$$\mathop{\mathbf{L}}\limits_{n} \mu'_n = \theta' \quad (\mathfrak{P};\ \mathfrak{S}); \qquad \mathop{\mathbf{L}}\limits_{n} \mu''_n = \theta'' \quad (\mathfrak{P};\ \mathfrak{S}),$$

imply

(3) $$a^{\mathfrak{A}} . \supset . \mathop{\mathbf{L}}\limits_{n} a\mu'_n = a\theta' \quad (\mathfrak{P};\ \mathfrak{S});$$

(4) $$\tau^{\mathfrak{T}} . \supset . \mathop{\mathbf{L}}\limits_{n} \tau\mu'_n = \tau\theta' \quad (\mathfrak{P};\ \mathfrak{T}\mathfrak{S});$$

(5) $$\mathfrak{S}^{D_1} . \supset . \mathop{\mathbf{L}}\limits_{n} (\mu'_n \pm \mu''_n) = \theta' \pm \theta'' \quad (\mathfrak{P};\ \mathfrak{S});$$

(6) $$\dagger . \mathfrak{S}^{D_1} . \supset . \mathop{\mathbf{L}}\limits_{n} \mu'_n\mu''_n = \theta'\theta'' \quad (\mathfrak{P};\ \mathfrak{S}\mathfrak{S}),$$

where in (6) the condition † is that $\theta'$ and $\theta''$ are dominated by $\mathfrak{A}\mathfrak{S}$.

More generally, from the relations:

$$\mathop{\mathbf{L}}\limits_{n} \mu'_n = \theta' \quad (\mathfrak{P}';\ \mathfrak{S}'); \qquad \mathop{\mathbf{L}}\limits_{n} \mu'' = \theta'' \quad (\mathfrak{P}'';\ \mathfrak{S}''),$$

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[*] *Eine neue Theorie der Convergenz und Divergenz von Reihen mit positiven Gliedern,* Crelle's Journal, vol. 76 (1873), pp. 61–91. Cf. p. 85.

[†] *Elementare Theorie der analytischen Functionen einer complexen Veränderlichen,* ed. 1 (1880), p. 22, § 25; ed. 2 (1898), p. 28, § 22.

[‡] *Sur les caractères de convergence des séries à terme positifs et sur les fonctions indéfiniment croissantes,* Acta Mathematica, vol. 18 (1894), pp. 319–336, p. 421. Cf. p. 321, theorem (b).

[§] Loc. cit., p. 328, theorem ($\beta$).

$\mu'$, $\theta'$, $\mathfrak{S}'$ being on $\mathfrak{P}'$; $\mu''$, $\theta''$, $\mathfrak{S}''$ being on $\mathfrak{P}''$, we have by § 13

$$(7)\quad \dagger' \,.\, \dagger'' \,.\, \mathfrak{S}'^{D_1} \,.\, \mathfrak{S}''^{D_1} \,.\, \supset . \underset{n}{\mathrm{L}}\, \mu'_n \mu''_n = \theta'\theta'' \quad (\mathfrak{P}'\mathfrak{P}''\,;\ \mathfrak{S}'\mathfrak{S}''),$$

where the conditions $\dagger'$, $\dagger''$ are that $\theta'$, $\theta''$ are dominated by $\mathfrak{A}\mathfrak{S}'$, $\mathfrak{A}\mathfrak{S}''$ respectively. $\mathfrak{S}'\mathfrak{S}''$ denotes the class of all functions $\sigma'\sigma''$ on $\mathfrak{P}'\mathfrak{P}'' \equiv [p'p'']$.

In case $\mathfrak{P}' = \mathfrak{P}'' = \mathfrak{P}$, a corollary of (7), by the assumption $p' = p'' = p$, is

$$(8)\quad \dagger' \,.\, \dagger'' \,.\, \mathfrak{S}'^{D_1} \,.\, \mathfrak{S}''^{D_1} \,.\, \supset . \underset{n}{\mathrm{L}}\, \mu'_n \mu''_n = \theta'\theta'' \quad (\mathfrak{P}\,;\ \mathfrak{S}'\mathfrak{S}'').$$

25. *Theorems.*—Of especial importance in the application of the property $D$ to the theory of double limits is the following

THEOREM I. *If a class $\mathfrak{S}^D$ is the scale class of each of a sequence of instances of relatively uniform convergence, then the corresponding sequence $\{\sigma_n\}$ of scale functions may be replaced by a single scale function $\sigma$ effective for each of the instances.*

Such a function $\sigma$ is the function $\sigma$ associated with the sequence $\{\sigma_n\}$ by the dominance property $D$ of the class $\mathfrak{S}$, the theorem being a consequence of § 9*a*1, 2, 3.

As such an application we notice the following

THEOREM II. *If the double array or sequence*

$$(1)\qquad \{\mu_{mn}\}$$

*of functions $\mu_{mn}$ on $\mathfrak{P}$ to $\mathfrak{A}$ gives rise to an iterated limit*

$$(2)\qquad \underset{m}{\mathrm{L}}\,\underset{n}{\mathrm{L}}\,\mu_{mn}$$

*convergent every inner limit and the outer limit uniformly as to a scale class $\mathfrak{S}$ having the dominance property $D$, that is, for a certain function $\eta$ and a sequence $\{\theta_m\}$ of functions $\theta_m$,*

$$(3)\quad m \,.\supset . \underset{n}{\mathrm{L}}\, \mu_{mn} = \theta_m \quad (\mathfrak{P}\,;\ \mathfrak{S})\,; \qquad \underset{m}{\mathrm{L}}\, \theta_m = \eta \quad (\mathfrak{P}\,;\ \mathfrak{S}),$$

*then there exist a function $\sigma$ and for every $m$ an integer $n_m$ and for every $e$ an integer $m_e$ such that*

$$(4)\qquad e \,.\, m \geqq m_e \,.\, n \geqq n_m \,.\supset .\, A(\mu_{mn} - \eta) \leqq eA\sigma,—$$

*a result expressible also in the form: for every $m$ there exists an integer $n_m$ such that*

(5) $$\underset{m}{\mathrm{L}}\, \mu_m = \eta \quad (\mathfrak{P}\,;\ \mathfrak{S})$$

*where for every $m$ $\mu_m$ denotes the many-valued function*

(6) $$\mu_m = \mu_{mn} \quad (n \geqq n_m),$$

*and in particular,*

(7) $$\underset{m}{\mathrm{L}}\, \mu_{mn_m} = \eta \quad (\mathfrak{P}\,;\ \mathfrak{S}).$$

*Proof.*—By theorem I, we have, from (3) and $\mathfrak{S}^D$, the existence of a function $\sigma$ of $\mathfrak{S}$ such that

(8) $$m \,.\supset .\, \underset{n}{\mathrm{L}}\, \mu_{mn} = \theta_m \quad (\mathfrak{P}\,;\ \sigma)\,; \qquad \underset{m}{\mathrm{L}}\, \theta_m = \eta \quad (\mathfrak{P}\,;\ \sigma).$$

From $(8_1)$ we have integers $n_{me}$ such that

(9) $$m \,.\, e \,.\, n \geqq n_{me} \,.\supset .\, A(\mu_{mn} - \theta_m) \leqq e A\sigma.$$

We take a sequence $\{e_m\}$ of positive numbers with

(10) $$\underset{m}{\mathrm{L}}\, e_m = 0,$$

and determine a sequence $\{n_m\}$ of positive integers by setting

(11) $$n_m = n_{me_m} \quad (m).$$

Then from (11, 9) we have

(12) $$m \,.\, n \geqq n_m \,.\supset .\, A(\mu_{mn} - \theta_m) \leqq e_m A\sigma.$$

Further, from $(8_2,\ 10)$ we have integers $m_e$ such that

(13) $$e \,.\, m \geqq m_e \,.\supset .\, 0 < e_m \leqq \frac{e}{2} \,.\, A(\theta_m - \eta) \leqq \frac{e}{2} A\sigma.$$

Then from (12, 13) we have

(14) $$e \,.\, m \geqq m_e \,.\, n \geqq n_m \,.\supset .\, A(\mu_{mn} - \eta) \leqq e A\sigma,$$

so that the hypotheses of the theorem make known the existence of $\sigma$, $\{n_m\}$, $(m_e)$ for which holds the relation (14) or (4) of the conclusion of the theorem.

Recalling the definition (§§ 14, 20) of $\mathfrak{M}_*$, the class $\mathfrak{M}$ extended as to $\mathfrak{S}$, as the class of all limit functions of sequences in $\mathfrak{M}$ convergent uniformly as to $\mathfrak{S}$, we have as a corollary of theorem II the important

THEOREM III. *The extension $\mathfrak{M}_{\mathfrak{S}}$ of a class $\mathfrak{M}$ as to a class $\mathfrak{S}$ having the dominance property $D$ is invariant under extension as to $\mathfrak{S}$, — in symbols:*

$$\mathfrak{M} \,.\, \mathfrak{S}^{D} \,.\supset.\, (\mathfrak{M}_{\mathfrak{S}})_{\mathfrak{S}} = \mathfrak{M}_{\mathfrak{S}} ;$$

*in particular,*

$$\mathfrak{M}^{D} \,.\supset.\, (\mathfrak{M}_{\mathfrak{M}})_{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}} .$$

For instance, since $\mathfrak{A}(\mathfrak{P})$ has the dominance property $D$,

$$\mathfrak{M} \,.\supset.\, (\mathfrak{M}_{\mathfrak{A}(\mathfrak{P})})_{\mathfrak{A}(\mathfrak{P})} = \mathfrak{M}_{\mathfrak{A}(\mathfrak{P})} ,$$

that is, the class $\mathfrak{N} \equiv \mathfrak{M}_{\mathfrak{A}(\mathfrak{P})}$ of all limits of uniformly convergent sequences of functions belonging to a class $\mathfrak{M}$ contains the limit of every uniformly convergent sequence of functions belonging to $\mathfrak{N}$.

Thus, if $\mathfrak{M}$ is the class of polynomials in $p$ on the linear interval $\mathfrak{P}^{\mathrm{IV}}$, the class $\mathfrak{N} \equiv \mathfrak{M}_{\mathfrak{A}(\mathfrak{P})}$ of all such limits is, according to the well-known theorem of WEIERSTRASS, the class $\mathfrak{M}^{\mathrm{IV}}$ of all continuous functions on $\mathfrak{P}^{\mathrm{IV}}$, for which the property specified is well known.

26. *Propositions.*—

(1) $$\mathfrak{M} \,.\, \mathfrak{S} \,.\supset.\, A(\mathfrak{M}_{\mathfrak{S}})^{(A\mathfrak{M})_{\mathfrak{S}}} .$$

(2) $$\mathfrak{M}^{A} \,.\, \mathfrak{S} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}}^{A} .$$

(3) $$\mathfrak{M}^{A} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}}^{A} .$$

(4) $$\mathfrak{M}^{L} \,.\, \mathfrak{S}^{D_1} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}}^{L} .$$

(5) $$\mathfrak{M}^{LD_1} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}}^{LD_1} .$$

(6) $$\mathfrak{M}^{AL} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}}^{ALD_1} .$$

(7) $$\mathfrak{M} \,.\, \mathfrak{S}^{\text{dominated by } \mathfrak{A}\mathfrak{T}} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}}^{\mathfrak{M}_{\mathfrak{T}}\,.\text{ dominated by } \mathfrak{M}_{\mathfrak{T}}} .$$

(8) $$\mathfrak{M}^{\text{dominated by } \mathfrak{A}\mathfrak{S}} \,.\, \mathfrak{S}^{D_1} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}}^{\text{dominated by } \mathfrak{A}\mathfrak{S}} .$$

(9) $$\mathfrak{M}^{D_1} \,.\, \mathfrak{S}^{\text{dominated by } \mathfrak{A}\mathfrak{M}} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}}^{\text{dominated by } \mathfrak{A}\mathfrak{M}} \,.\, \mathfrak{M}_{\mathfrak{S}}^{D_1} .$$

(10) $$\mathfrak{M}^{D} \,.\, \mathfrak{S}^{\text{dominated by } \mathfrak{A}\mathfrak{M}} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}}^{\text{dominated by } \mathfrak{A}\mathfrak{M}} \,.\, \mathfrak{M}_{\mathfrak{S}}^{D} .$$

(11) $$(A\mathfrak{M})^{\mathfrak{M}_{\mathfrak{S}}} \,.\, (\mathfrak{M}_{\mathfrak{S}})_{\mathfrak{S}} = \mathfrak{M}_{\mathfrak{S}} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}}^{A} .$$

(12) $$(A\mathfrak{M})^{\mathfrak{M}_{\mathfrak{S}}} \,.\, \mathfrak{S}^{D} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}}^{A} .$$

$$\mathfrak{S}^{D_1} \,.\supset.\, \mathfrak{S}_{\mathfrak{S}}^{\text{dominated by } \mathfrak{A}\mathfrak{S}}. \tag{13}$$

$$\mathfrak{M} \,.\, \mathfrak{S}^{D_1} \,.\supset.\, \mathfrak{M}_{(\mathfrak{S}\mathfrak{S})} = \mathfrak{M}_{\mathfrak{S}}. \tag{14}$$

$$\mathfrak{M} \,.\, \mathfrak{S}^{D} \,.\supset.\, (\mathfrak{M}_{\mathfrak{S}})_{\mathfrak{S}} = \mathfrak{M}_{\mathfrak{S}}. \tag{15}$$

$$\mathfrak{M} \,.\, \mathfrak{S}^{D} \,.\supset.\, \mathfrak{M}_{\mathfrak{S}} = (\mathfrak{M}_{\mathfrak{S}})_{\mathfrak{S}} = \mathfrak{M}_{(\mathfrak{S}\mathfrak{S})}. \tag{16}$$

$$\mathfrak{M}^{D_1} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}_{(\mathfrak{M}\mathfrak{M})} \,.\, (\mathfrak{M}_{\mathfrak{M}})_{\mathfrak{M}} = (\mathfrak{M}_{\mathfrak{M}})_{(\mathfrak{M}\mathfrak{M})}. \tag{17}$$

$$\mathfrak{M}^{D} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}} = \mathfrak{M}_{(\mathfrak{M}\mathfrak{M})} = (\mathfrak{M}_{\mathfrak{M}})_{\mathfrak{M}} = (\mathfrak{M}_{\mathfrak{M}})_{(\mathfrak{M}\mathfrak{M})}. \tag{18}$$

$$\mathfrak{M}^{D} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}}^{CD}. \tag{19}$$

$$\mathfrak{M}^{AD} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}}^{ACD}. \tag{20}$$

$$\mathfrak{M}^{LD} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}}^{LCD}. \tag{21}$$

$$\mathfrak{M}^{ALD} \,.\supset.\, \mathfrak{M}_{\mathfrak{M}}^{ALCD}. \tag{22}$$

Here the proof of $(10_2)$ that $\mathfrak{M}_{\mathfrak{S}}$ has the dominance property $D$ turns, with reference to $(10_1)$ and § 22*a*, on the considerations:

$$\mathfrak{M}^{\mathfrak{M}_{\mathfrak{S}}}; \quad \mathfrak{M}_{\mathfrak{S}}^{\text{dominated by } \mathfrak{A}\mathfrak{M}}; \quad \mathfrak{M}^{D} \,.\sim.\, (\mathfrak{A}\mathfrak{M})^{D}.$$

Further, (12) follows from (11) with the use of theorem III of § 25, which is restated here as (15).

We formulate (13, 19–22) as the following

THEOREM. *If a class $\mathfrak{M}$ has the dominance property $D$, then its extension $\mathfrak{M}_{\mathfrak{M}}$ as to itself is closed, is dominated by $\mathfrak{A}\mathfrak{M}$, and has the dominance property $D$; further, if $\mathfrak{M}$ is absolute, $\mathfrak{M}_{\mathfrak{M}}$ is absolute, and if $\mathfrak{M}$ is linear, $\mathfrak{M}_{\mathfrak{M}}$ is linear.*

### *Extension of classes of functions with respect to certain closure properties.* § 27.

27. *Extension of classes of functions.*—Theorem III of § 25 relates to instances in which the presence of the dominance property $D$ in the scale class ($\mathfrak{S}$ or $\mathfrak{M}$) of extension of a class $\mathfrak{M}$ implies in the extended class ($\mathfrak{M}_{\mathfrak{S}}$ or $\mathfrak{M}_{\mathfrak{M}}$) a closure property (closure as to $\mathfrak{S}$ or closure)—a property present in the original class $\mathfrak{M}$ only if it is invariant under the extension.

Now *closure as to a class* $\mathfrak{S}$ and *closure* are properties $P$ of classes $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$, such that (1°) the class $\overline{\mathfrak{M}}$ of all functions on $\mathfrak{P}$ to $\mathfrak{A}$ has the property $P$; (2°) the property $P$ belonging to the classes $\mathfrak{M}$ of a class $[\mathfrak{M}]$ of classes belongs also to the greatest common subclass $\cap[\mathfrak{M}]$ of the classes $\mathfrak{M}$; and accordingly (3°) for every class $\mathfrak{M}$ there exists a class $\mathfrak{M}_P$, and but one, containing $\mathfrak{M}$, having the property $P$ and contained in every class $\mathfrak{N}$ which contains $\mathfrak{M}$ and has the property $P$, viz., the greatest common subclass* of those classes $\mathfrak{N}$. This class $\mathfrak{M}_P$ is *the extension of* $\mathfrak{M}$ *as to the property* $P$. The property $P$ is, in the special sense indicated, *extensionally attainable.*

In the instances cited, the extension $\mathfrak{M}_P$ retains certain properties of $\mathfrak{M}$.

With respect to the properties $Q$ of $\mathfrak{M}$ retained, it would be of interest to compare with $\mathfrak{M}_P$ the various classes $\mathfrak{N}$ containing $\mathfrak{M}$ and having the property $P$. In certain cases it happens that $\mathfrak{M}_P$ retains a property $Q$ of $\mathfrak{M}$ retained by no other class $\mathfrak{N}$. Thus, in case $\mathfrak{M}$ is the class of polynomial functions of $p$ on the linear interval $\mathfrak{P} \equiv \mathfrak{P}^{IV}$, and $P$ is closure as to $\mathfrak{A}(\mathfrak{P})$, the extension $\mathfrak{M}_P$ is, with WEIERSTRASS, the class $\mathfrak{M}^{IV}$ of all continuous functions on $\mathfrak{M}^{IV}$; hence, the property $Q$ of $\mathfrak{M}$, that its functions are continuous on $\mathfrak{P}^{IV}$, is retained by the extension $\mathfrak{M}_P$, but by no other class $\mathfrak{N}$ (for instance, the class $\overline{\mathfrak{M}}$ of all functions on $\mathfrak{M}^{IV}$) containing $\mathfrak{M}$ and having the property $P$.

Any combination of properties extensionally attainable is extensionally attainable.

The properties:

$$A;\ L;\ C;\ C_{\mathfrak{S}},$$

viz., absolute; linear; closed; closed as to the class $\mathfrak{S}$, are extensionally attainable. Explicit formulas for the extensions:

$$\mathfrak{M}_A;\ \mathfrak{M}_L;\ \mathfrak{M}_{AL},$$

are given below (§§ 31, 43). Further, for brevity, setting for a class $\mathfrak{M}$:

$$(\mathfrak{M}_L)_{\mathfrak{M}} \equiv \mathfrak{M}_*;\qquad (\mathfrak{M}_{AL})_{\mathfrak{M}} \equiv \mathfrak{M}_\#,$$

the symbols $\mathfrak{M}_*$, $\mathfrak{M}_\#$ being read *the *-extension, the #-extension*

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* Necessarily existing.

*of the class* $\mathfrak{M}$, we have (§ 44), in case $\mathfrak{M}$ has the dominance property $D$, the formulas:

$$\mathfrak{M}_{LC} = \mathfrak{M}_{*}; \qquad \mathfrak{M}_{ALC} = \mathfrak{M}_{\#}.$$

These formulas with the additional items of the theorem of § 44 enter fundamentally in the theory, developed in Part II, of the composition of classes of functions of several variables. The reader so desiring may proceed directly to §§ 43, 44 and then to Part II.

## *Excursus.* §§ 28–42.

### *Extension of classes with respect to certain closure properties—in the theory of classes in general.*

28. *Closure properties in the theory of classes in general.*—One recognizes the characteristic rôle in mathematics of closure properties and of the process of extension of given classes to classes possessing specified properties. Accordingly in §§ 28–42 I develop a body of propositions of *the general theory of extension of classes with respect to closure properties*; these propositions are suggested by the special case of *the theory of classes of functions*; they belong however to *the theory of classes in general.*

We consider a *fundamental class* $\overline{\mathfrak{M}}$ *of elements* $\bar{\mu}$, its *subclasses* $\mathfrak{M}$, *properties* $P$ of classes $\mathfrak{M}$, and (single-valued) *transformations** $T$ on all classes $\mathfrak{M}$ to (not necessarily all) classes $\mathfrak{M}$.

The primary classes $\mathfrak{M}$ and properties $P$ about which statements are made are supposed to exist, to have extension, viz., the classes $\mathfrak{M}$ to contain one or more elements $\bar{\mu}$ of $\overline{\mathfrak{M}}$, and the properties $P$ to belong to one or more existent classes $\mathfrak{M}$. (Cf. the concluding remark of § 2.) The derivative classes and properties are not supposed to exist, being possibly the class or properties without extension, the null-class or null-properties, a null-property belonging to the null-class only.

We speak of the relations: *implication* ($\supset$); *equivalence* ($\sim$), between two *properties*: $P_1$; $P_2$, in case the corresponding implication ( . $\supset$ . ); equivalence ( . $\sim$ . ) holds generally for a system (in the present case, a class $\mathfrak{M}$) possessing the respective properties, viz.,

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* Functions on $[\mathfrak{M}]$ to $[\mathfrak{M}]$.

(1) $$P_1 \supset P_2 : \equiv : \mathfrak{M}^{P_1} . \supset . \mathfrak{M}^{P_2};$$

(2) $$P_1 \sim P_2 : \equiv : \mathfrak{M}^{P_1} . \sim . \mathfrak{M}^{P_2}.$$

Accordingly

(3) $$P_1 \sim P_2 : \sim : (P_1 \supset P_2) . (P_2 \supset P_1).$$

Implication is a transitive relation ; equivalence is a symmetric transitive relation.

We notice that

(4) $$(P_1 \sim P_1') . (P_2 \sim P_2') . (P_1 \supset P_2) . \supset . (P_1' \supset P_2'),$$

that is, the relation $\supset$ holding between two properties $P_1$, $P_2$ holds between two properties $P_1'$, $P_2'$ respectively equivalent to the properties $P_1$, $P_2$.

Similarly, the class $\mathfrak{M}_P$ (§ 29), the derived property $P_P$ (§ 29) and the transformation $T_P$ (§ 33) are defined, with reference to a property $P$, in such a way as to be respectively unchanged if the property $P$ is replaced by a property $P'$ equivalent to $P$.

Naturally, instead of considering in this way (classes of equivalent) properties $P$ of classes $\mathfrak{M}$, we might consider classes $P$ of classes $\mathfrak{M}$, a class $P$ consisting of those classes $\mathfrak{M}$ having the property $P$. Similarly, instead of considering properties (cf. § 30) of properties $P$ of classes $\mathfrak{M}$, we might consider classes of classes $P$ of classes $\mathfrak{M}$. On the other hand, instead of considering classes $\mathfrak{M}$ of elements $\bar{\mu}$ of the fundamental class $\bar{\mathfrak{M}}$, we might consider (classes of equivalent) properties $\mathfrak{M}$ of elements $\bar{\mu}$, a property $\mathfrak{M}$ belonging precisely to those elements $\bar{\mu}$ which belong to the class $\mathfrak{M}$; in this case, properties $P$ of classes $\mathfrak{M}$ are properties $P$ of properties $\mathfrak{M}$ (of elements $\bar{\mu}$). Similarly, properties (cf. § 30) of properties $P$ of classes $\mathfrak{M}$ are properties of properties $P$ of properties $\mathfrak{M}$ (of elements $\bar{\mu}$).

Thus the theory of the sequel is subject to modification of form. The form chosen is the form immediately available for the applications at present contemplated. However, the propositions being written symbolically may be read in the form of interpretation desired.

29. *The extension* $\mathfrak{M}_P$. *The derived property* $P_P$.—For a property $P$ and a class $\mathfrak{M}$ we define a class $\mathfrak{M}_P$, *the extension of* $\mathfrak{M}$ *as to* $P$,* as follows:

(1) $$\mathfrak{M}_P \equiv \bigcap [\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} \,.\, \mathfrak{N}^P],$$

viz., $\mathfrak{M}_P$ is the greatest common subclass of all classes $\mathfrak{N}$ (subclasses of $\overline{\mathfrak{M}}$) containing $\mathfrak{M}$ and having the property $P$.

Evidently, $\mathfrak{M}_P$ exists, if there exists any such class $\mathfrak{N}$, since every such class contains $\mathfrak{M}$.

Thus, if $\mathfrak{M}$ has the property $P$, $\mathfrak{M} = \mathfrak{M}_P$:

(2) $$\mathfrak{M}^P \,.\supset.\, \mathfrak{M} = \mathfrak{M}_P.$$

Further, a class $\mathfrak{M}$ having an extension $\mathfrak{M}_P$ has a second extension $(\mathfrak{M}_P)_P$ identical with the first:

(3) $$\mathfrak{M} \ni \exists\, \mathfrak{M}_P \,.\supset.\, \mathfrak{M}_P = (\mathfrak{M}_P)_P,$$

or, otherwise expressed,

(3′) $$\mathfrak{N} = \mathfrak{M}_P \,.\supset.\, \mathfrak{N} = \mathfrak{N}_P,$$

that is, the extension $\mathfrak{M}_P$ of a class $\mathfrak{M}$ is invariant under extension as to $P$.

For a property $P$, we define as the *derived property*† $P_P$, the property of *invariance under extension as to* $P$,—in symbols:

(4) $$\mathfrak{M}^{PP} \,.\equiv.\, \mathfrak{M} \ni \mathfrak{M} = \mathfrak{M}_P,$$

i. e.,

(4′) $$\mathfrak{M}^{P_P} \,.\equiv.\, \mathfrak{M} \ni \mathfrak{M} = \bigcap [\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} \,.\, \mathfrak{N}^P],$$

or, what is equivalent,

(4″) $$\mathfrak{M}^{P_P} \,.\equiv.\, \mathfrak{M} \ni \exists\, [\mathfrak{N}^P] \ni \mathfrak{M} = \bigcap [\mathfrak{N}].$$

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* This extension $\mathfrak{M}_P$ agrees, for properties $P$ extensionally attainable (cf. §§ 27, 30), with the extension $\mathfrak{M}_P$ of § 27. In general, however, the extension $\mathfrak{M}_P$ need not exist, and when existent need not have the property $P$.

† The property derived from a property $Q$ has the notation $P_Q$. Thus, $P{P_P}$ denotes the property derived from the property $P_P$, viz., from the property derived from a property $P$.

Then, from (3, 4),

$$(5) \qquad \mathfrak{M} \ni \exists\ \mathfrak{M}_P \,.\supset.\, \mathfrak{M}_P{}^{P_P}.$$

Further, from (2, 4), the property $P$ implies the derived property $P_P$:

$$(6) \qquad P \supset P_P,$$

viz.,

$$(6') \qquad \mathfrak{M}^{P} \,.\supset.\, \mathfrak{M}^{P_P}.$$

The class $P_P$: [all $\mathfrak{M}^{P_P}$], of all classes $\mathfrak{M}$ having the property $P_P$, includes the class $P$ : [all $\mathfrak{M}^{P}$], of all classes $\mathfrak{M}$ having the property $P$.

Since the extension $\mathfrak{M}_P$, quâ greatest common subclass $\cap[\mathfrak{N}]$ of the class $[\mathfrak{N}]$ of all classes $\mathfrak{N}$ containing $\mathfrak{M}$ and having the property $P$, is unchanged if the class $[\mathfrak{N}]$ is enlarged, we have the relation :

$$(7) \qquad \mathfrak{M} \,.\supset.\, \mathfrak{M}_P = \cap\,[\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} . \mathfrak{N} = \mathfrak{N}_P],$$

or, otherwise expressed,

$$(7') \qquad \mathfrak{M} \,.\supset.\, \mathfrak{M}_P = \cap\,[\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} . \mathfrak{N}^{P_P}] = \mathfrak{M}_{P_P}.$$

Thus, *a property* $P$ *and its derived property* $P_P$ (definable by (4″) without the mediation of the notion : extension) *give for every class* $\mathfrak{M}$ *the same extension:* $\mathfrak{M}_P = \mathfrak{M}_{P_P}$.

From (4, 6′) we see that an extensionally attainable property $P$ is equivalent to its derived property $P_P$:

$$(8) \qquad P^{\text{extensionally attainable}} \,.\supset.\, P \sim P_P.$$

Further, it appears * that the derived property $P_P$ is equivalent

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* From a class $P$ of classes $\mathfrak{M}$ we derive a class $P_\cap$ of classes $\mathfrak{M}$ by the greatest common subclass process $\cap$, viz., $P_\cap$ consists of the greatest common subclasses $\mathfrak{N}$ of the various classes $Q^P$ (subclasses of $P$) of classes $\mathfrak{M}$. Evidently the class $P_\cap$ contains the class $P$, and

$$(P_\cap)_\cap = P_\cap .$$

A class $P$ of classes $\mathfrak{M}$ has the property $\cap$, in notation : $P^\cap$, in case $P_\cap = P$. Thus, for every $P$,

$$P_\cap{}^\cap .$$

Consider a class $[P]$ of all classes $P$ with common class $P_\cap$ ; this common class $P_\cap$ belongs to the class $[P]$ and is the only class $P$ with the property $\cap$.

to the second derived property $P_{P_P}$:

$$(9) \qquad P . \supset . P_P \sim P_{P_P}.$$

30. *Properties of properties P.*—A property $P$ may have one or more of the following interrelated properties: †

$$(1) \qquad \overline{\mathfrak{M}}^P;$$

$$(2) \qquad \mathfrak{M} = \cap[\mathfrak{N}^P] . \supset . \mathfrak{M}^P;$$

$$(3) \qquad \mathfrak{M} . \supset . \exists\ \mathfrak{M}_P;$$

$$(4) \qquad \mathfrak{M} \ni \exists\ \mathfrak{M}_P . \supset . \mathfrak{M}_P{}^P.$$

We speak of properties $P^1$, $P^{34}$, etc.‡

A property $P^{34}$ is *extensionally attainable*, in the sense that for every class $\mathfrak{M}$ there exists a superclass $\mathfrak{M}_P$, the extension of $\mathfrak{M}$ as to $P$, having the property $P$ and contained in every superclass of $\mathfrak{M}$ with the property $P$. (Cf. § 27.)

A property $P^1$ is a property of the fundamental class $\overline{\mathfrak{M}}$.

*Every property $P^1$ is equivalent to a closure property*, in fact, to the closure $C_{T_0}$ (cf. § 34) as to the following transformation §

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In this notation, $P$ being the class of all classes $\mathfrak{M}^P$ having the property $P$, and $P_P$ being the class of all classes $\mathfrak{M}^{P_P}$ having the property $P_P$, by (4″) we have

$$P_P \quad P_\cap;$$

accordingly we have, in the terminology of classes,

$$P_{P_P} \quad (P_P)_\cap \quad (P_\cap)_\cap \quad P_\cap \quad P_P.$$

and hence, in the terminology of properties, as stated in the text,

$$P . \supset . P_P \sim P_{P_P}.$$

The theory of §§ 29–42 gains in perspicuity when considered in connection with the suggestions of this footnote and those of the closing paragraphs of § 28.

† 1) The fundamental class $\overline{\mathfrak{M}}$ has the property $P$.

2) If $\mathfrak{M}$ is the greatest common subclass $\cap[\mathfrak{N}]$ of a class $[\mathfrak{N}]$ of classes $\mathfrak{N}$ (subclasses of $\overline{\mathfrak{M}}$) having the property $P$, then $\mathfrak{M}$ itself has the property $P$.

3) For every class $\mathfrak{M}$ it is true that its extension $\mathfrak{M}_P$ exists.

4) For every class $\mathfrak{M}$ whose extension $\mathfrak{M}_P$ exists it is true that the extension $\mathfrak{M}_P$ has the property $P$.

‡ E. g., a property $P^2$ is a property $P$ with the property 2.

§ This transformation $T_0$ is cited below in §§ 38.1₂, 39.14, 41.

$T_0$: $T_0\mathfrak{M} = \mathfrak{M}$ or $\overline{\mathfrak{M}}$, according as $\mathfrak{M}$ does or does not have the property $P$.

31. *Functiontheoretic examples.*—Within a fundamental class $\overline{\mathfrak{M}}^P$ of functions the properties $P$:

$$P = A\,;\ L\,;\ \textit{multiplicative}\,;\ C_{\mathfrak{S}}\,;\ C,$$

have the properties 1, 2 and so the properties 3, 4. They are extensionally attainable. As to the corresponding extensions $\mathfrak{M}_P$ of a class $\mathfrak{M}$ we notice:

(1) $\mathfrak{M}_A$ is the least common superclass of $\mathfrak{M}$ and $A\mathfrak{M}$, viz., the class of all functions of the form:

$$\mu \quad \text{or} \quad A\mu.$$

(2) $\mathfrak{M}_L$ is the class of all functions of the form:

$$\sum_{i=1}^{i=n} a_i\mu_i,$$

viz., linear homogeneous in a finite number of functions of $\mathfrak{M}$ with coefficients from $\mathfrak{A}$.

(3) $\mathfrak{M}_{\text{multiplicative}}$ is the class of all functions of the form:

$$\prod_{i=1}^{i=n} \mu_i,$$

viz., the product of a finite number of functions of $\mathfrak{M}$, repetitions allowed.

(4) $$\mathfrak{M}\,.\,\mathfrak{S}^D\,.\supset.\,\mathfrak{M}_{C_{\mathfrak{S}}} = \mathfrak{M}_{\mathfrak{S}}.$$

(5) $$\mathfrak{M}^D\,.\supset.\,\mathfrak{M}_C = \mathfrak{M}_{\mathfrak{M}}.$$

As to (4, 5) cf. theorem III of § 25.

32. *Propositions concerning properties $P$ and their derived properties $P_P$.*—

(1) $$P^1\,.\sim.\,P^3,^* \qquad P^2\,.\sim.\,P^4\,;$$
$$P^{\text{extensionally attainable}}\,.\sim.\,P^{12}\,.\sim.\,P^{14}\,.\sim.\,P^{32}\,.\sim.\,P^{34}\,.\sim.\,P^{1234}.$$

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[*] This ( . ∼ . ) is the equivalence of the two *statements*: that a property $P$ has the property 1 (§ 30) and that it has the property 3. The equivalence ($\sim$) of two *properties* (cf. § 28) occurs, e. g., in (16, 24, 25).

(2) $$\mathfrak{M} \ni \exists\, \mathfrak{N} \ni (\mathfrak{M}^{\mathfrak{N}} . \mathfrak{N}^{P}) . \supset . \exists\, \mathfrak{M}_P .$$

(3) $$\mathfrak{M} \ni \exists\, \mathfrak{M}_P . \supset . \mathfrak{M}^{\mathfrak{M}_P} . (\mathfrak{M}_P)_P = \mathfrak{M}_P .$$

(4) $$\mathfrak{M} \ni \exists\, \mathfrak{M}_P : \supset : \mathfrak{M}_P = \mathfrak{M} . \sim . \mathfrak{M}_P{}^{\mathfrak{M}} . \sim . \mathfrak{M} \ni \exists\, \mathfrak{M}' \ni \mathfrak{M}'_P = \mathfrak{M}.$$

(5) $$\mathfrak{M} = \cap\, [\mathfrak{N}^{P}] . \supset . \mathfrak{M}_P = \mathfrak{M}.$$

(6) $$\mathfrak{M}^{P_P} . \sim . \mathfrak{M}_P = \mathfrak{M} . \sim . \mathfrak{M} \ni (\exists\, \mathfrak{M}_P . \mathfrak{M}_P{}^{\mathfrak{M}}) . \sim . \mathfrak{M} \ni \exists\, \mathfrak{M}' \ni \mathfrak{M}'_P = \mathfrak{M}.$$

(7) $$\mathfrak{M} \ni \exists\, \mathfrak{M}_P . \supset . \mathfrak{M}_P{}^{P_P} .$$

(8) $$\mathfrak{M}^{P} . \supset . \mathfrak{M}_P = \mathfrak{M} . \mathfrak{M}^{P_P} ; \qquad P \supset P_P .$$

(9) $$\mathfrak{M} . \supset . \mathfrak{M}_P = \cap\, [\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} . \mathfrak{N}^{P_P}],$$

that is,

(9′) $$\mathfrak{M} . \supset . \mathfrak{M}_P = \cap\, [\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} . \mathfrak{N}_P = \mathfrak{N}].^*$$

(10) $$\mathfrak{M} . \supset . \mathfrak{M}_P = \mathfrak{M}_{P_P} .$$

(11) $$\mathfrak{M}^{P_{P_P}} . \sim . \mathfrak{M}^{P_P} . \sim . \mathfrak{M}_P = \mathfrak{M} ; \qquad P_{P_P} \sim P_P .$$

(12)† $$P_P \supset P : \sim : \mathfrak{M}^{P_P} . \supset . \mathfrak{M}^{P} : \sim : \mathfrak{M} = \cap\, [\mathfrak{N}^{P}] . \supset . \mathfrak{M}^{P},$$

so that, by $(8_2)$ and § 30.2,

(13)‡ $$P^2 : \sim : P_P \sim P : \sim : \mathfrak{M}^{P_P} . \sim . \mathfrak{M}^{P} .$$

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* For $\mathfrak{M}_P$ and $\cap[\mathfrak{N}]$ exist simultaneously, and, when existent, $\mathfrak{M}_P$ is one of the classes $\mathfrak{N}$ while every class $\mathfrak{N}$ contains $\mathfrak{M}_P$, and accordingly $\mathfrak{M}_P = \cap[\mathfrak{N}]$.

† As to a property $P$ the three statements:

(1°) The derived property $P_P$ implies the property $P$;
(2°) Every class $\mathfrak{M}$ having the derived property $P_P$ has the property $P$;
(3°) The greatest common subclass $\cap[\mathfrak{N}]$ of a class $[\mathfrak{N}]$ of classes $\mathfrak{N}$ having the property $P$ has the property $P$,

are equivalent.

(1°) and (2°) are equivalent by the definition of implication ($\supset$) between properties (cf. § 28). (2°) and (3°) are equivalent by the definition of derived properties (§ 29).

‡ As to a property $P$ the three statements:

(1°) The property $P$ has the property 2 of § 30;
(2°) The property $P$ is equivalent to its derived property $P_P$;
(3°) Every class $\mathfrak{M}$ having the property $P$ has the derived property $P_P$; and conversely,

are equivalent.

(14) $$\mathfrak{M} = \cap\,[\mathfrak{N}^{P_P}] \,.\supset.\, \mathfrak{M}^{P_P},$$
that is, by (6),

(14′) $$\mathfrak{M} = \cap\,[\mathfrak{N} \ni \mathfrak{N}_P = \mathfrak{N}] \,.\supset.\, \mathfrak{M}_P = \mathfrak{M}.^{*}$$

(15) $$P \,.\supset.\, (P \supset P_P)\,.\,P_P{}^2.^{\dagger}$$

(16) $$P^2 \,.\sim.\, (P_P \sim P) \,.\sim.\, (P_P \supset P).$$

(17)‡ $$P^1 :\supset: P_P{}^{1234} : \mathfrak{M} \,.\supset.\, \exists\, \mathfrak{M}_P \,.\, \mathfrak{M}^{\mathfrak{M}_P} \,.\, \mathfrak{M}_P = (\mathfrak{M}_P)_P = \mathfrak{M}_{P_P}.$$

(18) $$P^1 :\supset: \mathfrak{M}_1{}^{\mathfrak{M}_2} \,.\supset.\, \mathfrak{M}_{1P}{}^{\mathfrak{M}_{2P}}.$$

(19) $$P^1 :\supset: \mathfrak{M}_1{}^{\mathfrak{M}_2} \,.\, \mathfrak{M}_{2P}{}^{\mathfrak{M}_2} \,.\supset.\, \mathfrak{M}_{1P}{}^{\mathfrak{M}_2}.$$

(20) $$P^1 :\supset: \mathfrak{M}_1{}^{\mathfrak{M}_2} \,.\, \mathfrak{M}_2{}^{P} \,.\supset.\, \mathfrak{M}_{1P}{}^{\mathfrak{M}_2}.$$

(21)§ $$P_1{}^1 \,.\, P_2{}^1 :\supset: \mathfrak{M}_{P_1}{}^{P_2} \,.\supset.\, \mathfrak{M}_{P_2}{}^{\mathfrak{M}_{P_1}}.$$

(22)‖ $$P_1{}^1 \supset P_2{}^1 :\supset: \mathfrak{M} \,.\supset.\, \mathfrak{M}_{P_2}{}^{\mathfrak{M}_{P_1}}.$$

(23) $$P_1 \supset P_2 \,.\supset.\, P_{P_1} \supset P_{P_2}.$$

(24) $$P_1 \sim P_2 \,.\supset.\, P_{P_1} \sim P_{P_2}.$$

(25) $$P_1{}^1 \sim P_2{}^1 :\supset: \mathfrak{M} \,.\supset.\, \mathfrak{M}_{P_1} = \mathfrak{M}_{P_2}.$$

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[*] We have
$$\mathfrak{M} = \cap\,[\mathfrak{N} \ni \mathfrak{N} = \cap\,[\text{all } \mathfrak{N}' \ni \mathfrak{N}^{\mathfrak{N}'} \,.\, \mathfrak{N}'^{P}]],$$
$$\mathfrak{M}_P = \cap\,[\text{all } \mathfrak{N}'' \ni \mathfrak{M}^{\mathfrak{N}''} \,.\, \mathfrak{N}''^{P}].$$

$\mathfrak{M}$, the $\mathfrak{N}$, the $\mathfrak{N}'$ for every $\mathfrak{N}$ are supposed to exist. The classes $\mathfrak{N}'$ contain $\mathfrak{M}$ and have the property $P$ and are accordingly amongst the classes $\mathfrak{N}''$. Thus $\mathfrak{M}_P$ exists and is contained in every class $\mathfrak{N}'$ and so in every class $\mathfrak{N}$ and so in $\mathfrak{M} = \cap\,[\mathfrak{N}]$. On the other hand, by $(3_1)$, since $\mathfrak{M}_P$ exists, $\mathfrak{M}$ is contained in $\mathfrak{M}_P$. Accordingly, under the hypothesis, $\mathfrak{M}_P = \mathfrak{M}$.

[†] $(15_1)$ is $(8_2)$. A comparison of (11, 13) yields $(15_2)$.

[‡] If a property $P$ has the property 1 of § 30, of belonging to the fundamental class $\mathfrak{M}$, then the derived property $P_P$ is extensionally attainable and for every class $\mathfrak{M}$ the extension $\mathfrak{M}_P$ exists and contains $\mathfrak{M}$ and is identical with the second extension $(\mathfrak{M}_P)_P$ and also with the extension $\mathfrak{M}_{P_P}$ as to the derived property $P_P$.

[§] If $P_1$ and $P_2$ are two properties having the property 1 of § 30, then, if the (necessarily existing) extension $\mathfrak{M}_{P_1}$ of a class $\mathfrak{M}$ as to the property $P_1$ has the property $P_2$, it contains the (necessarily existing) extension $\mathfrak{M}_{P_2}$ as to the property $P_2$.

[‖] If of two properties $P_1$, $P_2$ having the property 1 of § 30 the first implies the second, then for every class $\mathfrak{M}$ the extension as to the first property contains the extension as to the second property.

33. *The transformation $T_P$ for a property $P^1$.*—The properties $P^1$: of the fundamental class $\overline{\mathfrak{M}}$, and the properties $P^3$: for every $\mathfrak{M}$ the extension $\mathfrak{M}_P$ exists, are equivalent (§ 32.1$_1$). For a property $P^1$ *extension as to $P$* is *the transformation $T_P$* transforming a class $\mathfrak{M}$ into $\mathfrak{M}_P$, the class $\mathfrak{M}$ extended as to $P$, viz.,

$$\mathfrak{M} \,.\supset.\, T_P\mathfrak{M} = \mathfrak{M}_P .$$

34. *The closure $C_T$.*—For a transformation $T$ *closure as to $T$* is the property $C_T$:

(1) $$\mathfrak{M}^{C_T} .\equiv.\, T\mathfrak{M}^{\mathfrak{M}},$$

viz., a class $\mathfrak{M}$ is closed as to $T$ in case it contains its transform $T\mathfrak{M}$ by $T$. Similarly, *extensibility by $T$*, *invariance under $T$* are the properties $E_T$, $I_T$:

(2) $$\mathfrak{M}^{E_T} .\equiv.\, \mathfrak{M}^{T\mathfrak{M}},$$

(3) $$\mathfrak{M}^{I_T} .\equiv.\, T\mathfrak{M} = \mathfrak{M},$$

so that $I_T$ is equivalent to the composite property $C_TE_T$, viz.,

(4) $$T :\supset: I_T \sim C_TE_T .$$

35. *The extension $\mathfrak{M}_T$. The derived transformation $T_T$.*—The fundamental class $\overline{\mathfrak{M}}$ is closed as to every transformation $T$, so that (by § 32.1$_1$) for every transformation $T$ and class $\mathfrak{M}$ the extension $\mathfrak{M}_{C_T}$ of $\mathfrak{M}$ as to the closure $C_T$ exists. For brevity we set

(1) $$\mathfrak{M}_{C_T} \equiv \mathfrak{M}_T,$$

(2) $$T_{C_T} \equiv T_T,$$

so that (cf. §§ 29, 33, 34)

(3) $$\mathfrak{M} \,.\supset.\, \mathfrak{M}_T = \bigcap\,[\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} \,.\, T\mathfrak{N}^{\mathfrak{N}}],$$

(4) $$\mathfrak{M} \,.\supset.\, T_T\mathfrak{M} = \mathfrak{M}_T .$$

Thus, corresponding to a transformation $T$ we have *the derived transformation*[*] $T_T$ and for every class $\mathfrak{M}$ *the extension $\mathfrak{M}_T$ of $\mathfrak{M}$ as to $T$.*

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[*] The transformation derived from a transformation $U$ has the notation $T_U$. Thus, $T_{T_T}$ denotes the transformation derived from the transformation $T_T$, viz., from the transformation derived from a transformation $T$.

36. *Propositions concerning properties* $P^1$. — For a property $P^1$: of the fundamental class $\overline{\mathfrak{M}}$, we have, by §§ 29–35,

(1) $$C_{T_P} \sim P_P\,;$$

(2) $$\mathfrak{M}^{P_P} . \sim . T_P\mathfrak{M}^{\mathfrak{M}} . \sim . \mathfrak{M}_P{}^{\mathfrak{M}} . \sim . \mathfrak{M}_P = \mathfrak{M}\,;$$

(3) $$\mathfrak{M} . \supset . (\mathfrak{M}_P)_P = \mathfrak{M}_P = \mathfrak{M}_{P_P} = \mathfrak{M}_{C_{T_P}} = \mathfrak{M}_{T_P},$$

and, accordingly,

(4) $$T . \supset . P_{C_T} \sim C_{T_{C_T}} \sim C_{T_T}\,;$$

(5) $$P^1 : \supset : \mathfrak{M} . \supset . (\mathfrak{M}_P)_P = \mathfrak{M}_P = \mathfrak{M}_{P_P} = \cap\,[\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} . \mathfrak{N}_P{}^{\mathfrak{N}}].$$

37. *Properties of transformations* $T$. — A transformation $T$ may have one or more of the following interrelated properties:

(1) $$\mathfrak{M} . \supset . \mathfrak{M}^{T\mathfrak{M}}.$$

(2) $$\mathfrak{M} . \supset . TT\mathfrak{M}^{T\mathfrak{M}}.$$

(3) $$\mathfrak{M} . \supset . TT\mathfrak{M} = T\mathfrak{M}.$$

(4) $$\mathfrak{M} . \supset . \mathfrak{M}_T{}^{T\mathfrak{M}}.$$

(5) $$\mathfrak{M} . \supset . T\mathfrak{M}^{\mathfrak{M}_T}.$$

(6) $$T = T_T\,,$$

that is,

(6′) $$\mathfrak{M} . \supset . T\mathfrak{M} = T_T\mathfrak{M} = \mathfrak{M}_T.$$

(7) $$\mathfrak{M}_1{}^{\mathfrak{M}_2} . \supset . T\mathfrak{M}_1{}^{T\mathfrak{M}_2}.$$

(8) $$\mathfrak{M}_1{}^{\mathfrak{M}_2} . T\mathfrak{M}_2{}^{\mathfrak{M}_2} . \supset . T\mathfrak{M}_1{}^{\mathfrak{M}_2}.$$

($Q$) $$\mathfrak{M} . \supset . T\mathfrak{M} = \cup\,[\text{all } T\mathfrak{M}_0 \ni \mathfrak{M}_0{}^{\mathfrak{M}.Q}],$$

that is, with respect to a property $Q$ of classes $\mathfrak{M}$, a transformation $T$ has the associated property $Q$ in case for every class $\mathfrak{M}$ the transform $T\mathfrak{M}$ is the least common superclass of the transforms $T\mathfrak{M}_0$ of the various classes $\mathfrak{M}_0$, subclasses of $\mathfrak{M}$, having the property $Q$. The property $Q$ of classes $\mathfrak{M}$ is supposed to be such that every class $\mathfrak{M}$ has one or more subclasses $\mathfrak{M}_0$ with the property $Q$; accordingly, $Q$ is a property of every element $\bar{\mu}$ quâ class $\mathfrak{M}$. As instances we have the following properties:

$Q$: *singular* (consisting of a single element);

*finite* (consisting of a finite number of elements);

*denumerable* (finite or consisting of a denumerable infinitude of elements); in general,

*of cardinal number less than* or *of cardinal number not exceeding a given cardinal number.*

We speak of transformations $T^1$, $T^{124}$, $T^Q$, etc. Transformations $T^1$ are *extensional*. Transformations $T^7$ *leave "contains" invariant*. Transformations $T^Q$ are (or have the property $Q$) *associated with the property $Q$ of classes $\mathfrak{M}$.*

Evidently the transitivity of *contains* implies that every transformation $T^Q$ leaves *contains* invariant:

(9) $$T^Q \,.\supset .\, T^7.$$

Every transformation $T$ induces a transformation $T'$ on single elements (quâ classes) to classes $\mathfrak{M}$. Conversely, any such transformation $T'$ induces a transformation $T^{\text{singular}}$ of which it is the induced transformation $T'$. — Further, every transformation $T^{\text{singular}}$ is a transformation $T^{\text{finite, denumerable}}$.

37*a*. *Functiontheoretic examples of transformations $T^{Q785}$.* — On respectively suitable fundamental classes $\overline{\mathfrak{M}}$ of functions the seven transformations $T$:

$$T\mathfrak{M} = A\mathfrak{M}\,;\ \mathfrak{A}\mathfrak{M}\,;\ \mathfrak{M} + \mathfrak{M}\,;\ \mathfrak{A}\mathfrak{M} + \mathfrak{A}\mathfrak{M}\,;\ \mathfrak{M}\mathfrak{M}\,;\ \mathfrak{M}\ \ ;\ \mathfrak{M}_{w},$$

respectively, are transformations $T^7$: leaving *contains* invariant, and accordingly (cf. § 39.7, 5) they are transformations $T^{578}$. The classes $\mathfrak{M}$ closed under these transformations $T$ respectively are the classes $\mathfrak{M}$ containing $T\mathfrak{M}$. Hence (§ 15) the corresponding closure properties $C_T$ of classes $\mathfrak{M}$ are the properties:

$C_T = A$ ; *multiplicative as to* $\mathfrak{A}$ ;
*additive* ; $L$ ; *multiplicative* ; $C_{.}$ ; $C$.

Of these transformations the second, fourth, sixth, seventh are transformations $T^1$: extensional, so that for them closure as to $T$ and invariance under $T$ are equivalent properties of classes $\mathfrak{M}$.

Further, the seven transformations are transformations $T^{Q}$, —the first two for $Q$: *singular;* the next three for $Q$: *finite, or consisting of one or two elements;* the last two for $Q$: *denumerable.*

38. *Propositions concerning the transformations* $T_{P^1}$.—

(1) [*] $P^{1} \,.\supset .\, T_{P}^{12345678} \,.\, \exists\, T^{1234} \ni (C_{T} \sim P \,.\, T_{T} = T_{P})$.

(2) $P_1^{1} \,.\, P_2^{1} \therefore \supset \therefore$

$$T_{P_1} = T_{P_2} : \sim : C_{T_{P_1}} \sim C_{T_{P_2}} : \sim : P_{P_1} \sim P_{P_2} : \sim :$$
$$\mathfrak{M} \,.\supset .\, \mathfrak{M}_{P_1} = \mathfrak{M}_{P_2} : \sim : \mathfrak{M}_{P_1}{}^{\mathfrak{M}} \,.\sim .\, \mathfrak{M}_{P_2}{}^{\mathfrak{M}}$$
$$: \sim : \mathfrak{M}_{P_1} = \mathfrak{M} \,.\sim .\, \mathfrak{M}_{P_2} = \mathfrak{M}.$$

39. *Propositions concerning transformations* $T$.—

(1) $T^{12} \,.\supset .\, T^{34}$; [†] $T^{3} \,.\supset .\, T^{2}$; $T^{13} \,.\supset .\, T^{24}$.

(2) $T : \supset : \mathfrak{M} \,.\supset .\, \mathfrak{M}^{\mathfrak{M}_T}$.

(3) $T^{4} \,.\supset .\, T^{1}$.

(4) $T^{12} \,.\sim .\, T^{13} \,.\sim .\, T^{24} \,.\sim .\, T^{34} \,.\sim .\, T^{1234}$.

(5) $T^{5} \,.\sim .\, T^{8}$.

(6) $T^{48} \,.\sim .\, T^{45} \,.\sim .\, T^{6}$.

(7) $T^{7} \,.\supset .\, T^{8}$.

(8) $T^{128} \,.\supset .\, T^{7}$.

(9) $T \,.\supset .\, C_T^{1} \,.\, T_T^{12345678}$ [§] $.\; T_T = T_{C_T} = T_{C_{T_T}} = T_{T_T}$. [||]

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[*] For every property $P$ having the property 1 (of § 30) the derived transformation $T_P$ (cf. § 33) has the properties 1, 2, 3, 4, 5, 6, 7, 8 (of § 37) and there exists a transformation $T$ having the properties 1, 2, 3, 4 (of § 37) whose closure $C_T$ (cf. § 34) is equivalent to the property $P$ and whose derived transformation $T_T$ (cf. § 35) is the derived transformation $T_P$.

For the first conclusion, cf. § 36 and § 32.17, 18, 19.

For the second conclusion, the transformation $T_0$ defined at the end of § 30 is a suitable instance.

[†] $\mathfrak{M}_T = \cap$ [all $\mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} \,.\, T\mathfrak{N}^{\mathfrak{N}}$] by definition. By hypothesis, $T^{12}$, so that $\mathfrak{M} \,.\supset .\, \mathfrak{M}^{T\mathfrak{M}} \,.\, TT\mathfrak{M}^{T\mathfrak{M}}$, that is, for every $\mathfrak{M}$ $T\mathfrak{M}$ is one of the classes $\mathfrak{N}$ whose greatest common subclass is $\mathfrak{M}_T$. Hence $\mathfrak{M} \,.\supset .\, \mathfrak{M}_T{}^{T\mathfrak{M}}$, that is, $P^{4}$.

[§] Cf. §§ 39.$9_1$, 35.2, 38.1.

[||] We prove that

$$\mathfrak{M} \,.\supset .\, T_T\mathfrak{M} = T_{T_T}\mathfrak{M}.$$

For we have

(10) $$T^1 . \supset . C_T \sim I_T .$$

(11) $$T^8 . \supset . C_T^{1234} . C_T \sim P_{C_T} \sim C_{T_T} .$$*

(12) $$T^6 . \supset . T^{12345678} .$$†

(13) $$T^{127} . \supset . T^{128} . \supset . T^{125} . \supset . T^{16} .$$

(14)‡ $$T^{125} . \sim . T^{127} . \sim . T^{128} . \sim . T^6 . \sim . T^{12345678} .$$

$$T_T \mathfrak{M} \quad \mathfrak{M}_T \quad \cap [\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} . T\mathfrak{N}^{\mathfrak{N}}];$$

$$T_{T_T} \mathfrak{M} \quad \mathfrak{M}_{T_T} \quad \cap [\text{all } \mathfrak{N}_1 \ni \mathfrak{M}^{\mathfrak{N}_1} . T_T \mathfrak{N}_1^{\mathfrak{N}_1}]$$

$$\cap [\text{all } \mathfrak{N}_1 \ni \mathfrak{M}^{\mathfrak{N}_1} . \mathfrak{N}_{1T}^{\mathfrak{N}_1}].$$

Now every $\mathfrak{N}$ is closed as to $T$, so that $\mathfrak{N}_T \quad \mathfrak{N}$. Thus every $\mathfrak{N}$ is an $\mathfrak{N}_1$. Hence $T_T \mathfrak{M} \quad \cap [\mathfrak{N}]$ contains $T_{T_T} \mathfrak{M} \quad \cap [\mathfrak{N}_1]$.

Further, since by $(9_2)$ $T_T$ has the property 8 of § 37, $(\mathfrak{M}^{\mathfrak{N}_1} . T_T \mathfrak{N}_1^{\mathfrak{N}_1})$ implies $T_T \mathfrak{M}^{\mathfrak{N}_1}$. Thus $T_T \mathfrak{M}$ is contained in every $\mathfrak{N}_1$ and hence in $T_{T_T} \mathfrak{M}_1 \quad \cap [\mathfrak{N}_1]$.

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* Cf. §§ 39.11$_1$, 32.16, 36.4.

† As to $T^2$ cf. §§ 37.6, 35.1, 39.9$_1$, 32.1, 32.3$_2$.

‡ The six triads:

125 ; 127 ; 128 ; 135 ; 137 ; 138,

of properties are equivalent (§ 39.14, 1$_2$).

Further, each triad is a triad of independent properties. This appears from the following instances of transformations.

Let $\overline{\mathfrak{M}}$ be a linear class of functions containing a function $p \neq 0(\mathfrak{P})$. The three transformations:

$$T_1 : T_1 \mathfrak{M} \quad p;$$

$$T_2 : T_2 \mathfrak{M} \quad \cup(\mathfrak{M}, 2\mathfrak{M});$$

$$T_3 : T_3 \mathfrak{M} \quad \mathfrak{M} \ (\mathfrak{M} \neq p), \qquad T_3 p \quad \overline{\mathfrak{M}},$$

have respectively

$$\mathfrak{M}_T \quad \cup(\mathfrak{M}, p); \quad \cap [\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} . 2\mathfrak{M}^{\mathfrak{N}}]; \quad \mathfrak{M},$$

and

$$\mathfrak{M}^{C_T} . \sim . p^{\mathfrak{N}}; \quad 2\mathfrak{M}^{\mathfrak{N}}; \quad \mathfrak{M} \neq p.$$

For the three transformations as to the eight properties (1 8) the table is as follows:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $T_1$: | — | + | \| | — | \| | | \| | \| |
| $T_2$: | \| | | | — | \| | — | \| | \| |
| $T_3$: | \| | \| | \| | \| | — | | | |

,

where the tabular entry is | or — according as the transformation has or has not the property.

Instead of those three transformations one might consider these three: the first and the fourth of § 37$a$ and that of § 30, the respective parameters: $\overline{\mathfrak{M}}$; $\overline{\mathfrak{M}}$; $\overline{\mathfrak{M}}$, $P$, being suitably determined.

(15) $$T_1 \,.\, T_2 \,.\, (C_{T_1} \sim C_{T_2}) \,.\, \supset \,.\, T_{T_1} = T_{T_2} \,.$$

(16) $$T_1^6 \,.\, T_2^6 \,.\, (C_{T_1} \sim C_{T_2}) \,.\, \supset \,.\, T_1 = T_2 \,.$$

(17) $$T_1^6 \,.\, T_2^6 : \supset : (I_{T_1} \sim I_{T_2}) \,.\, \sim \,.\, T_1 = T_2 \,.^*$$

From (11, 9, 12, 16) we have

THEOREM I. *The transformations* $T^8$ *being classified with respect to their closures* $C_T$, *— two transformations* $T_1^8$, $T_2^8$ *with equivalent closures:* $C_{T_1} \sim C_{T_2}$, *being of the same class, — in every class there is precisely one transformation* $T^{12345678}$, *viz., in the class containing a transformation* $T^8$ *the corresponding derived transformation* $T_T$.

For example, one such class consists of all transformations † $T$ such that every class $\mathfrak{M}$ is closed as to $T$; the corresponding unique transformation is the transformation identity, leaving every class invariant.

Further, from (10, 11, 9, 12, 17) we have

THEOREM II. *The transformations* $T^{18}$ *being classified with respect to their systems of invariant classes, i. e., with respect to their invariance properties* $I_T$ *or with respect to their closures* $C_T$, *in every class there is precisely one transformation* $T^{12345678}$, *viz., in the class containing a transformation* $T^{18}$ *the corresponding derived transformation* $T_T$.

The classes of transformations $T^{18}$ are, in fact, contained in the classes of transformations $T^8$. The systems of classes $\mathfrak{M}$ closed as to the $T^8$ and the systems closed as to or invariant under the $T^{18}$ are identical, being the various classes $P^{1234}$ ‡ of classes $\mathfrak{M}$, viz., the classes $P$ containing the fundamental class $\overline{\mathfrak{M}}$ and having the greatest common subclass property $\cap$ (cf. the last footnote of § 29).

THEOREM III. *Under a transformation* $T^7$*: leaving "contains" invariant, the transform* $T\mathfrak{M}'$ *of the* $\substack{\textit{least common superclass}\\ \textit{greatest common subclass}}$ $\mathfrak{M}'$ *of a class* $[\mathfrak{M}]$ *of classes* $\mathfrak{M}$ $\substack{\textit{contains}\\ \textit{is contained in}}$ *the* $\substack{\textit{least common superclass}\\ \textit{greatest common subclass}}$ *of the transforms* $T\mathfrak{M}$ *of the classes* $\mathfrak{M}$.

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* Cf. (12, 10).

† To this class of transformations belong the ZERMELO transformations, viz., transformations on classes $\mathfrak{M}$ to elements $\bar{\mu}$, quâ classes, every class $\mathfrak{M}$ transforming into one of its own elements $\bar{\mu}$.

‡ Corresponding to the properties $P^{1234}$ : extensionally attainable.

The relations *least common superclass* and *greatest common subclass* are in general not invariant,[*] in fact not even under the transformations $T^{127}$.

We have, from §§ 32.1, 38.1, 36.1, 32.16,

$$P^{34}\,.\supset .\,P^{1234}\,.\supset .\,T_P^{12345678}\,.\supset .\,C_{T_P}\sim P_P\sim P,$$

and, from §§ 39.13, 39.12, 39.11, 35.2, 37.6,

$$T^{127}\,.\supset .\,T^{6}\,.\supset .\,T^{12345678}\,.\supset .\,C_T^{1234}\,.\supset .\,T_{C_T}=T_T=T,$$

and accordingly

THEOREM IV. *The correspondences:*

*to every property* $P^{1234}$ *the derived transformation* $T_P$*;*

*to every transformation* $T^{12345678}$ *the invariance property* $I_T$
*or its equivalent the closure property* $C_T$,

*constitute a one-one correspondence between the classes of equivalent properties* $P^{1234}$ *(the properties extensionally attainable) and the transformations* $T^{12345678}$ *of such a nature that for every class* $\mathfrak{M}$*, possession of a property* $P$ *and invariance under the corresponding transformation* $T$ *are equivalent.*

39*a*. *Propositions*[†] *concerning properties* $P$ *and transformations* $T$.—

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[*] Consider the transformation $T^{127} = T_{\text{additive}}$ (cf. §§ 37*a*, 33, 35.2, 39.9) and two non-zero functions $\mu_1$, $\mu_2$ on $\mathfrak{P}$ between which there is identically in $p$ no relation of the form $n_1\mu_1 + n_2\mu_2 = 0$, the notations $n$ denoting as usual positive integers.

Taking the classes

we have $\mathfrak{M}_1 = \mu_1,\quad \mathfrak{M}_2 = \mu_2,\quad \mathfrak{M}_3 = [\mu_1, \mu_2],\quad \mathfrak{M}_4 = [\mu_1, 2\mu_2]$

$$T\mathfrak{M}_1 = [\text{all } n\mu_1],\qquad T\mathfrak{M}_2 = [\text{all } n\mu_2],$$

$$T\mathfrak{M}_3 = [\text{all } n\mu_1,\ n\mu_2,\ n_1\mu_1 + n_2\mu_2],\qquad T\mathfrak{M}_4 = [\text{all } n\mu_1,\ 2n\mu_2,\ n_1\mu_1 + 2n_2\mu_2].$$

$\mathfrak{M}_3$ is the least common superclass of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, while $\mathfrak{M}_1$ is the greatest common subclass of $\mathfrak{M}_3$ and $\mathfrak{M}_4$. $T\mathfrak{M}_3$ contains but is not the least common superclass [all $n\mu_1$, $n\mu_2$] of $T\mathfrak{M}_1$ and $T\mathfrak{M}_2$. $T\mathfrak{M}_1$ is contained in but is not the greatest common subclass [all $n\mu_1$, $2n\mu_2$, $n_1\mu_1 + 2n_2\mu_2$] of $T\mathfrak{M}_3$ and $T\mathfrak{M}_4$.

[†] 1) If of two properties: $P_1$; $P_2$, having the property 1 of § 30 the first implies the second, then for every class $\mathfrak{M}$ the (necessarily existing) extension as to the first property contains the (necessarily existing) extension as to the second property.

2) For two properties: $P_1$; $P_2$, having the property 1 of § 30, the rela

(1) $$P_1^1 \supset P_2^1 : \supset : \mathfrak{M} . \supset . \mathfrak{M}_{P_2}{}^{\mathfrak{M}P_1} .$$

(2) $$P_1 \supset P_{P_2} : \subset : \mathfrak{M} . \supset . \mathfrak{M}_{P_2^1}{}^{\mathfrak{M}P_1^1} .$$

(3) $$P_1^1 \supset P_2^{12} : \sim : \mathfrak{M} . \supset . \mathfrak{M}_{P_2^{12}}{}^{\mathfrak{M}P_1^1} .$$

(4) $$P_1^1 \sim P_2^1 : \supset : \mathfrak{M} . \supset . \mathfrak{M}_{P_1} = \mathfrak{M}_{P_2} .$$

(5) $$P_1 \sim P_2 : \subset : \mathfrak{M} . \supset . \mathfrak{M}_{P_1^{12}} = \mathfrak{M}_{P_2^{12}} .$$

(6) $$P_1^{12} \sim P_2^{12} : \sim : \mathfrak{M} . \supset . \mathfrak{M}_{P_1^{12}} = \mathfrak{M}_{P_2^{12}} .$$

From these propositions (in view of § 39.9, 5, 11 ; § 35.1 ; § 36.4) we obtain others involving $C_T$, $\mathfrak{M}_T$, and then, using the conditions 4, 5, 6 of § 37 to replace $\mathfrak{M}_T$ by $T\mathfrak{M}$, we have the following propositions :

(7) $$P^1 \supset C_{T^5} : \supset : \mathfrak{M} . \supset . T\mathfrak{M}^{\mathfrak{M}P} .$$

(8) $$P \supset C_{T_T} : \subset : \mathfrak{M} . \supset . T^4\mathfrak{M}^{\mathfrak{M}P^1} .$$

(9) $$P^1 \supset C_{T^6} : \sim : \mathfrak{M} . \supset . T^6\mathfrak{M}^{\mathfrak{M}P^1} .$$

(10) $$C_{T^4} \supset P^1 : \supset : \mathfrak{M} . \supset . \mathfrak{M}_P{}^{T'\mathfrak{M}} .$$

(11) $$C_T \supset P_P : \subset : \mathfrak{M} . \supset . \mathfrak{M}_{P^1}{}^{T^5\mathfrak{M}} .$$

(12) $$C_{T^6} \supset P^{12} : \sim : \mathfrak{M} . \supset . \mathfrak{M}_{P^{12}}{}^{T^6\mathfrak{M}} .$$

(13) $$P^{12} \sim C_{T^6} : \sim : \mathfrak{M} . \supset . \mathfrak{M}_{P^{12}} = T^6\mathfrak{M} .$$

(14) $$C_{T_1^4} \supset C_{T_2^5} : \supset : \mathfrak{M} . \supset . T_2\mathfrak{M}^{T_1\mathfrak{M}} .$$

(15) $$C_{T_1} \supset C_{T_{T_2}} : \subset : \mathfrak{M} . \supset . T_2^4\mathfrak{M}^{T_1^5\mathfrak{M}} .$$

(16) $$C_{T_1^6} \supset C_{T_2^6} : \sim : \mathfrak{M} . \supset . T_2^6\mathfrak{M}^{T_1^6\mathfrak{M}} .$$

---

tion that the first implies the property derived from the second is implied by the relation that for every class $\mathfrak{M}$ the extension as to the first contains the extension as to the second.

3) For two properties: $P_1$ ; $P_2$, the first having the property 1 and the second the properties : 1 ; 2, of § 30, the two relations : the first implies the second ; for every class $\mathfrak{M}$ the extension as to the first contains the extension as to the second, are equivalent relations.

⋮

21) $T$ being a transformation having the property 6 of § 37, for every class $\mathfrak{M}$ the transform $T\mathfrak{M}$ is the least common superclass of the extensions $\mathfrak{M}_P$ of the class $\mathfrak{M}$ as to the properties $P$ which have the property 1 of § 30 and are implied by the closure as to $T$.

$$(17) \quad C_{T_1^6} \sim C_{T_2^6} : \sim : \mathfrak{M} \,.\supset.\, T_1^6\mathfrak{M} = T_2^6\mathfrak{M} : \sim : T_1^6 = T_2^6.$$

From (7, 10), in view of the properties of $T_P$; $P_P$; $\mathfrak{M}_T$; $T_T$, we have the propositions:

$$(18)^* \quad P^1 : \supset : \mathfrak{M} \,.\supset.\, \mathfrak{M}_P = \cup \,[\text{all } T\mathfrak{M} \ni T^5 \,.\, (P \supset C_T)].$$

$$(19) \quad P^{12} : \supset : \mathfrak{M} \,.\supset.\, \mathfrak{M}_P = \cap \,[\text{all } T\mathfrak{M} \ni T^1 \,.\, (C_T \supset P)].$$

$$(20)^* \quad T^6 : \supset : \mathfrak{M} \,.\supset.\, T\mathfrak{M} = \cap \,[\text{all } \mathfrak{M}_P \ni P^1 \,.\, (P \supset C_T)].$$

$$(21)^* \quad T^6 : \supset : \mathfrak{M} \,.\supset.\, T\mathfrak{M} = \cup \,[\text{all } \mathfrak{M}_P \ni P^1 \,.\, (C_T \supset P)].$$

40. *Composition of properties $P$.* — As explained in § 3, any class $[P]$ of properties $P$ gives rise to a *composite property*† $\cap[P]$, the logical product of the properties of the class $[P]$, so that for a class $\mathfrak{M}$ the two statements:

$$\mathfrak{M}^{\cap[P]}; \qquad P^{[P]} \,.\supset.\, \mathfrak{M}^P,$$

are equivalent.

The properties $P$ of $[P]$ are *mutually consistent* and the composite property $\cap[P]$ is *with extension*, in case there exists a class $\mathfrak{M}$ with the composite property $\cap[P]$.

Obviously, equivalence of properties is invariant under composition of properties.

40*a*. *Propositions concerning composite properties.* —

$$(1) \qquad [P^i] \,.\supset.\, \cap[P]^i \qquad (i = 1, 2, 3, 4),$$

that is, a class $[P^i]$ of properties $P^i$ with the property $i$ $(i = 1, 2, 3, 4)$ of § 30 gives rise to a composite property $\cap[P]$ with the property $i$. Every composite property $\cap[P^1]$, quâ a property of $\overline{\mathfrak{M}}$, is with extension. We have, as a corollary,

$$(2) \qquad [P^{12}] \,.\supset.\, \cap[P]^{12},$$

and accordingly, by § 32.1, the

THEOREM. *A class $[P]$ of properties $P$ extensionally attainable gives rise to a composite property $\cap[P]$ itself extensionally*

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[*] In (18; 20; 21) we may replace $(T^5;\ P^1;\ P^1)$ by $(T^7;\ P^{12};\ P^{12})$.

[†] In case the class $[P]$ consists of a finite number of properties: $P = P_1; \cdots; P_n$, we denote, as heretofore, the composite property $\cap\ [P]$ also by the notation $P_1P_2 \cdots P_n$.

*attainable; the class* $[P]$ *is a class of mutually consistent properties.*

The properties $P_1$, $P_2$ being extensionally attainable, the composite property $P_1P_2$ is likewise extensionally attainable. For every class $\mathfrak{M}$ the three notations:

$$\mathfrak{M}_{P_1P_2} \quad (\equiv \mathfrak{M}_{(P_1P_2)}); \qquad (\mathfrak{M}_{P_1})_{P_2}; \qquad (\mathfrak{M}_{P_2})_{P_1},$$

denote three classes, extensions of $\mathfrak{M}$ definite and in general distinct. In the notation $\mathfrak{M}_{P_1P_2}$, as in $\mathfrak{M}^{P_1P_2}$, the interchange of symbols $P_1$, $P_2$ does not alter the meaning.

41. *Composition of transformations* $T$.—Any class:

$$\mathfrak{T} = [T],$$

of transformations $T$ gives rise to a *composite transformation:*

$$T_{\mathfrak{T}},$$

viz., the transformation* $T_{\mathfrak{T}}$ for which for every class $\mathfrak{M}$

$$T_{\mathfrak{T}}\mathfrak{M} = \mathbf{U}\,[\text{all } T\mathfrak{M} \ni T^{\mathfrak{T}}],$$

that is, the composite transformation $T_{\mathfrak{T}}$ transforms the class $\mathfrak{M}$ into the least common superclass of the transforms $T\mathfrak{M}$ of the class $\mathfrak{M}$ by the transformations $T$ of the class $\mathfrak{T}$.

For instance, consider a property $P^1$: of the fundamental class $\overline{\mathfrak{M}}$. The class $\mathfrak{T}$ of all transformations $T$ (or $T^1$: extensional) with closure $C_T$ equivalent to $P$ contains the transformation† $T_0 : T_0\mathfrak{M} = \mathfrak{M}$ or $\overline{\mathfrak{M}}$ according as $\mathfrak{M}$ has or has not the property $P$, and this transformation $T_0$ is the composite transformation $T_{\mathfrak{T}}$ of the class $\mathfrak{T}$.

41*a*. *Propositions concerning composite transformations.*—

(1) $$\mathfrak{T} = [T] \,.\supset.\, C_{T_{\mathfrak{T}}} \sim \boldsymbol{\cap}\,[\text{all } C_T \ni T^{\mathfrak{T}}],$$

viz., for every class $\mathfrak{T}$ of transformations $T$ closure as to the composite transformation $T_{\mathfrak{T}}$ is equivalent to the composite of the closures as to the transformations $T$ of the class $\mathfrak{T}$, that is,

---

*In case the class $\mathfrak{T}$ consists of a single transformation $T$, the composite transformation $T_{\mathfrak{T}}$ is the transformation $T$ itself; it is not the derived transformation $T_T$ defined in § 35.

† Cf. §§ 30, $38.1_2$, 39.14.

(1′) $\mathfrak{T} = [T] . \mathfrak{M} \therefore \supset \therefore T\mathfrak{M}^{m} : \sim : T^{i} . \supset . T\mathfrak{M}^{m}$.

(2)[†] $\mathfrak{T} = [T^{1}] . \supset . T_{\mathfrak{T}}^{1}$.

(3) $\mathfrak{T} = [T^{7}] . \supset . T_{\mathfrak{T}}^{7}$.

($Q$) $\mathfrak{T} = [T^{Q}] . \supset . T_{\mathfrak{T}}^{Q}$.

Thus, the properties 1 : extensional ; 7 : leaving *contains* invariant ; $Q$ : associated with the property $Q$ of classes $\mathfrak{M}$, of § 37 are invariant under composition of transformations. Evidently composites of these properties are similarly invariant.

As a corollary of (1) we have the

THEOREM. *The transformations $T$ being classified with respect to their closures, composition of transformations induces a corresponding composition of classes of transformations. A similar statement holds for the similar classification of transformations $T^{i}$, $i$ being a property invariant under composition of transformations, for instance, 1, 7, $Q$ or any of their composites.*

42. *Multiplication of transformations $T$. Semigroups.* In the theory of *transformations in general*, two transformations $T_1$, $T_2$ by *multiplication* give rise to the *product transformation* $T_2T_1$ such that for every class $\mathfrak{M}$ $(T_2T_1)\mathfrak{M} = T_2(T_1\mathfrak{M})$. This multiplication is associative and usually non-commutative. The *powers* of a transformation $T$ are denoted by $T \equiv T^{(1)}$; $TT \equiv T^{(2)}$; $\cdots$; $T^{(n)}$, $\cdots$, where $T^{(n+1)} \equiv TT^{(n)}$.

A class :

$$\mathfrak{T} = [T],$$

of transformations $T$ gives rise to or *generates* the *product class :*

$$\mathfrak{T}' = [\text{all } T'],$$

consisting of all transformations $T'$ of the form :

$$T' = T_nT_{n-1} \cdots T_1,$$

viz., the product of a finite number : $n \geqq 1$, of transformations $T$ of the class $\mathfrak{T}$.

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[†] A class $\mathfrak{T} = [T^{1}]$ of transformations $T^{1}$ : extensional, gives rise to a composite transformation $T_{\mathfrak{T}}$ with the same property.

The product class $\mathfrak{T}'$ contains the class $\mathfrak{T}$. If $\mathfrak{T}' = \mathfrak{T}$, the class $\mathfrak{T}$ is a *semigroup*. Otherwise expressed, the class $\mathfrak{T}$ is a semigroup in case it is *closed under multiplication*, that is, in case the product of every two transformations $T$ of the class $\mathfrak{T}$ belongs to $\mathfrak{T}$. For every class $\mathfrak{T}$ the corresponding class $\mathfrak{T}'$ is a semigroup.

The class $\bar{\mathfrak{T}}$ of all transformations $T$ is a semigroup; and the greatest common subclass of a class of semigroups is a semigroup. Thus, by § 32.1, the property of being a semigroup is extensionally attainable, in the sense of § 30, the class $\mathfrak{M}$ being here the class $\mathfrak{T}$. A class $\mathfrak{T}$ has as its extension as to this property the corresponding class $\mathfrak{T}'$.

A transformation $T$, quâ singular class $\mathfrak{T}$, generates a *cyclic semigroup* $\mathfrak{T}'$, viz., the class of all powers of $T$.

In the present case of transformations $T$ of classes $\mathfrak{M}$, the subclasses of a fundamental class $\bar{\mathfrak{M}}$, there are certain relations between the processes of composition and of multiplication of transformations $T$.

42*a*. *Propositions concerning multiplication and composition of transformations $T$.*—

(1) $$T_1^1 \,.\, T_2^1 \,.\supset.\, (T_2 T_1)^1 \,.\, C_{T_2 T_1} \sim (C_{T_1} C_{T_2}) \sim C_{T_1 T_2} .$$

(2) $$T_1^7 \,.\, T_2^7 \,.\supset.\, (T_2 T_1)^7 \,.\, (C_{T_1} C_{T_2}) \supset C_{T_2 T_1} .$$

(3) $$T_1^Q \,.\, T_2^Q \,.\supset.\, (T_2 T_1)^Q \qquad (Q = \textit{singular, finite, denumerable}),$$

that is, the specified properties $Q$ of transformations (associated with the corresponding properties $Q$ of classes $\mathfrak{M}$, cf. § 37) are invariant under multiplication of transformations.

$$\left.\begin{array}{ll} (4) & \mathfrak{T} = [T^i] : \supset : T'^{\,\mathfrak{T}'} . \supset . T'^{\,i} \\ (5) & \mathfrak{T} = [T^i] \,.\supset.\, T_{\mathfrak{T}'}^{\;i} \end{array}\right\} \quad \begin{array}{r} (i = 1,\ 7,\ \textit{singular, finite, denumerable}, \\ \text{or a composite of these properties}), \end{array}$$

that is, a class $\mathfrak{T}$ of transformations $T^i$ with specified property $i$ generates a semigroup, whose transformations $T'$ and composite transformation $T_{\mathfrak{T}'}$ have the property $i$.

(6) $$\mathfrak{T} = [T^1] \,.\supset.\, C_{T_{\mathfrak{T}}} \sim [\text{all } C_T \ni T^{\mathfrak{T}}] \sim C_{T_{\mathfrak{T}'}} .$$

As a corollary, comparable with the theorem of § 41$a$, we have

THEOREM I. *The transformations $T^1$, viz., the extensional transformations (or $T^{17}$, viz., the extensional transformations leaving "contains" invariant) being classified as to their closures, multiplication of transformations induces a corresponding multiplication of classes of transformations. This multiplication of classes is commutative as well as associative. Each class contains its composite.*[*]

*Such a class of transformations $T^{17}$, viz., of extensional transformations leaving "contains" invariant, is a class of transformations $T^{178}$ (§ 39.7) with closure extensionally attainable (§ 39.11) containing precisely one $T^{12345678}$, viz., for every $T$ the derived transformation $T_t$ (§ 39, theorem II). This transformation $T^{12345678}$ is the composite of the class.*

In proof of the last remark, denoting by $T_0$ the composite of the class of transformations including a transformation $T_1^{17}$, we are to prove that

$$T_0 = T_{t_1},$$

that is, for every class $\mathfrak{M}$:

$$T_{t_1}\mathfrak{M} = \mathfrak{M}_{t_1} = T_0\mathfrak{M},$$

where

$$\mathfrak{M}_{t_1} = \bigcap\,[\text{all } \mathfrak{N} \ni \mathfrak{M}^{\mathfrak{N}} \,.\, T_1\mathfrak{N}^{\mathfrak{N}}],$$

$$T_0\mathfrak{M} = \bigcup\,[\text{all } T\mathfrak{M} \ni T^{17} \,.\, (C_T \sim C_{T_1})].$$

In the first place, by § 39.11, $T_{t_1}$ is such a transformation $T$, so that $T_0\mathfrak{M}$ contains $T_{t_1}\mathfrak{M} = \mathfrak{M}_{t_1}$. In the second place, $\mathfrak{M}_{t_1}$ contains $T_0\mathfrak{M}$, for every such $\mathfrak{N}$ contains every such $T\mathfrak{M}$; in fact, $(\mathfrak{M}^{\mathfrak{N}} \,.\, T^{\mathfrak{N}})$ implies $T\mathfrak{M}^{T\mathfrak{N}}$; and $(T_1\mathfrak{N}^{\mathfrak{N}} \,.\, C_T \sim C_{T_1})$ implies $T\mathfrak{N}^{\mathfrak{N}}$; so that, as stated, $T\mathfrak{M}^{\mathfrak{N}}$.

The following theorem concerns cases in which for a transformation $T$ the corresponding transformation $T_t$ is the composite of the cyclic semigroup of $T$.

THEOREM II. *The transformation $T$ being extensional and associated with $Q$: singular, or finite, and accordingly*[†] *leaving*

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[*] The class of these composites is not necessarily closed under multiplication.

[†] Cf. § 37.9.

*"contains" invariant* (§ 37.1, $Q$, 7), *its powers* $T^{(n)}$ *and the composite* $T_0$ *of its cyclic semigroup:* $[\text{all } T^{(n)}]$, *viz., the transformation* $T_0$ *for which for every class* $\mathfrak{M}$

$$(7) \qquad T_0\mathfrak{M} \equiv \mathbf{U}\ [\text{all } T^{(n)}\mathfrak{M}],$$

*have the same three properties and their closures are equivalent* (§ 42*a* 1–6). *Moreover,*

$$(8) \qquad T_0T = T_0 = TT_0 = T_0T_0.$$

*Thus,* $T_0$ *has the property* 2 *of* § 37. *Hence:* *

$$(9) \qquad T_0^{12345678}; \qquad T_{T'} = T_{T_0} = T_0.$$

We proceed to the proof of (8) for the case $Q$: finite, the proof for the case $Q$: singular, being even simpler.

$T$ being extensional, $T^{(n)}\mathfrak{M}$ is always contained in $T^{(n+1)}\mathfrak{M}$, so that, in view of (7), in an obvious sense,

$$(10) \qquad T_0 = \mathop{\mathbf{L}}_{n} T^{(n)}, \qquad T_0\mathfrak{M} = \mathop{\mathbf{L}}_{n} T^{(n)}\mathfrak{M}.$$

Accordingly, for every class $\mathfrak{M}$, $T_0T\mathfrak{M} = T_0\mathfrak{M}$; that is, $T_0T = T_0$.

Further, $T_0\mathfrak{M}$ is always contained in $TT_0\mathfrak{M}$. On the other hand (cf. § 37 $Q$), $T$ being associated with $Q$: finite, the class:

$$TT_0\mathfrak{M} = \mathbf{U}\ [\text{all } T\mathfrak{N} \ni \mathfrak{N}^{T_0\mathfrak{M}\,.\,\text{finite}}],$$

is contained in $T_0\mathfrak{M}$. In fact, such a finite subclass $\mathfrak{N}$ of $T_0\mathfrak{M}$ is a finite subclass $\mathfrak{N}$ of some $T^{(n)}\mathfrak{M}$, and accordingly every $T\mathfrak{N}$ is a subclass of some $T^{(n+1)}\mathfrak{M}$ and so of $T_0\mathfrak{M}$. Hence, always $T_0\mathfrak{M} = TT_0\mathfrak{M}$; that is, $T_0 = TT_0$.

We have, for every $n$ and $\mathfrak{M}$, $T_0\mathfrak{M} = T^{(n)}T_0\mathfrak{M}$, and so, by (7), $T_0\mathfrak{M} = T_0T_0\mathfrak{M}$; that is, $T_0 = T_0T_0$.

After this excursus (§§ 28–42) on the theory of classes in general, we return to the consideration of classes $\mathfrak{M}$ of functions on a general class $\mathfrak{P}$ to the class $\mathfrak{A}$ of real numbers.

### *The extensions:*

### $\mathfrak{M}_{AL}$; $\quad \mathfrak{M}_* \equiv (\mathfrak{M}_L)_{\mathfrak{M}}$; $\quad \mathfrak{M}_\# \equiv (\mathfrak{M}_{AL})_{\mathfrak{M}}$,

### *of classes* $\mathfrak{M}$ *of functions of a general variable.* §§ 43–44.

43. *The extension* $\mathfrak{M}_{AL}$.—The properties: $A \equiv$ absolute,

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* In view of § 39.14 and § 37.6, by theorem I.

$L$ = linear, and their composite: $AL$, are, as we saw in § 31 and § 10$a$, extensionally attainable within the class $\overline{\mathfrak{M}}$ of all functions on $\mathfrak{P}$ to $\mathfrak{A}$. The formulas for the corresponding extensions $\mathfrak{M}_A$, $\mathfrak{M}_L$ of a class $\mathfrak{M}$ were given in § 31. In terms of the corresponding extensional transformations $T_A$, $T_L$:

$$(1) \qquad T_A\mathfrak{M} = \mathfrak{M}_A, \qquad T_L\mathfrak{M} = \mathfrak{M}_L,$$

which are associated with $Q$: finite (§ 37 $Q$), and their product $T_L T_A$:

$$(2) \qquad (T_L T_A)\mathfrak{M} = T_L(T_A\mathfrak{M}) = (\mathfrak{M}_A)_L,$$

in accordance with § 42$a$II, for the extension $\mathfrak{M}_{AL}$ we have the formula:

$$(3) \qquad \mathfrak{M}_{AL} = \mathbf{U}\,[\text{all } (T_L T_A)^{(n)}\mathfrak{M}],$$

which is expressible also in the forms:

$$(4) \qquad \mathfrak{M}_{AL} = \mathbf{L}_n (T_L T_A)^{(n)}\mathfrak{M} = \mathbf{L}_n (T_A T_L)^{(n)}\mathfrak{M}.$$

The formula (3) states that the class $\mathfrak{M}_{AL}$, i. e., the class $\mathfrak{M}$ extended to be absolute and linear (cf. §§ 27, 29), consists of all the functions each of which belongs to some class $(T_L T_A)^{(n)}\mathfrak{M}$,* i. e., to the class arising from the class $\mathfrak{M}$ by the transformation $T_L T_A$ repeated $n$ times. The transformation $T_L T_A$ transforms any class $\mathfrak{M}$ into the class: $T_L T_A\mathfrak{M}$, consisting of all functions of the form:

$$\sum_{i=1}^{i=k} a_i\mu_j + \sum_{j=k+1}^{j=l} a_j A\mu_j .$$

It is convenient to write

$$(5) \quad \mathfrak{M} \equiv [\mu], \qquad \mathfrak{M}_A \equiv [\mu_A], \qquad \mathfrak{M}_L \equiv [\mu_L], \qquad \mathfrak{M}_{AL} \equiv [\mu_{AL}].$$

Then $\mu_A$ is of the form: $\mu$ or $A\mu$, and $\mu_L$ is of the form:

$$(6) \qquad \mu_L = \sum_{i=1}^{i=n} a_i\mu_i ,$$

while $\mu_{AL}$ is derived from a finite number of functions $\mu$ of $\mathfrak{M}$ by a combination of operations $T_A$, $T_L$ involving a finite number of coefficients $a$ of $\mathfrak{A}$.

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* I. e., $\mathfrak{M}_{AL}$ is the least common superclass of the classes $(T_L T_A)^{(n)}\mathfrak{M}$ (cf. § 2).

44. *The $*$- and $\#$-extensions $\mathfrak{M}_*$, $\mathfrak{M}_\#$.*—For a class $\mathfrak{M}$ we defined in § 27 the *$*$-extension* $\mathfrak{M}_*$ and the *$\#$-extension* $\mathfrak{M}_\#$:

$$(1) \qquad (\mathfrak{M}_L)_{\mathfrak{M}} \equiv \mathfrak{M}_* \equiv [\mu_*]; \qquad (\mathfrak{M}_{AL})_{\mathfrak{M}} \equiv \mathfrak{M}_\# \equiv [\mu_\#].$$

Accordingly $\mu_*$, $\mu_\#$ are of the respective forms :[*]

---

[*] A function $\mu_*$ is the limit of a sequence $\{\mu_{Ln}\}$ of functions $\mu_{Ln}$ of the class $\mathfrak{M}_L$, which converges on $\mathfrak{P}$ uniformly as to the class $\mathfrak{M}$, and accordingly, as to some function $\mu$ of the class $\mathfrak{M}$. A function $\mu_{Ln}$ is linear homogeneous in a finite number of functions of $\mathfrak{M}$, the coefficients being real numbers. Arranging the totality of functions of $\mathfrak{M}$ thus related to the functions of the sequence $\{\mu_{Ln}\}$ as a sequence $\{\mu_n\}$, we suppose that the function $\mu_{Ln}$ is linear homogeneous in the first $m_n$ functions $\mu_k$ with coefficients $a_{nk}$ $(k=1, 2, \cdots, m_n)$, of which some or all may vanish.

Hence, we may set

$$(1) \qquad \mu_* = \mathop{\mathrm{L}}_n \sum_{k=1}^{m_n} a_{nk}\,\mu_k \quad (\mathfrak{P}\,;\,\mu).$$

In particular, for the case :

$$(2) \qquad m_n = n \quad (n)\,; \qquad a_{nk} = 1 \quad (nk),$$

we see that $\mathfrak{M}_*$ contains the class of all functions $\mu_*$ of the form :

$$(3) \qquad \mu_* = \mathop{\mathrm{L}}_n \sigma_n = \sum_{n=1}^{\infty} \mu_n \quad (\mathfrak{P}\,;\,\mu),$$

where

$$(4) \qquad \sigma_n = \sum_{k=1}^{n} \mu_k \quad (n),$$

viz., of all sums of series :

$$(5) \qquad \mu_* = \sum_{n=1}^{\infty} \mu_n \quad (\mathfrak{P}\,;\,\mu),$$

converging on $\mathfrak{P}$ uniformly as to $\mathfrak{M}$.

Further, the general case (1) may be looked at as of form (5), the convergence however being a *modified convergence* regulated by the system :

$$(6) \qquad \{m_n\}\,; \qquad \left(a_{nk} \,\middle|\, \begin{matrix} n=1, 2, \cdots\cdots \\ k=1, 2, \cdots, m_n \end{matrix}\right),$$

of integers $m_n$ and coefficients $a_{nk}$. Thus, for the case :

$$(7) \qquad m_n = n \quad (n)\,; \qquad a_{nk} = \frac{n-k+1}{n} \quad (nk),$$

so that by (4)

$$(8) \qquad \sum_{k=1}^{m_n} a_{nk}\mu_k = \frac{1}{n}\sum_{k=1}^{n} \sigma_k,$$

the convergence is the HÖLDER convergence of the sequence of arithmetical means of the $n$ first partial sums $\sigma_k$ for $n=1, 2, 3, \cdots$.

In recent researches on divergent series (e. g., power series ; Fourier series) various types of modified convergence have played a central rôle.

$$\mu_* = \mathop{\mathrm{L}}_{n} \mu_{Ln} = \mathop{\mathrm{L}}_{n} \sum_{k=1}^{m_n} a_{\ k}\mu_k \quad (\mathfrak{P};\ \mu);$$
$$\mu_\sharp = \mathop{\mathrm{L}}_{n} \mu_{ALn} \quad (\mathfrak{P};\ \mu). \tag{2}$$

The corresponding transformations: $T_*$; $T_\sharp$, viz., for every $\mathfrak{M}$

$$T_* \mathfrak{M} = \mathfrak{M}_*; \qquad T_\sharp \mathfrak{M} = \mathfrak{M}_\sharp, \tag{3}$$

are extensional transformations leaving *contains* invariant. Classes $\mathfrak{M}$ closed as to $T_*$, $T_\sharp$ respectively are invariant under $T_*$, $T_\sharp$. For the corresponding closure properties we have the equivalences:

$$C_{T_*} \sim LC \sim \text{(linear, closed)};$$
$$C_{T_\sharp} \sim ALC \sim \text{(absolute, linear, closed)}. \tag{4}$$

These closures are extensionally attainable, but they are in general not attained in $\mathfrak{M}_*$, $\mathfrak{M}_\sharp$. However, *for classes $\mathfrak{M}$ with the dominance property $D$, the closures are in fact attained in* $\mathfrak{M}_*$, $\mathfrak{M}_\sharp$ (cf. § 44*a*).

44*a*. *Propositions concerning the various classes* $\mathfrak{N}$:

$$\mathfrak{N}:\ \mathfrak{M};\ \mathfrak{M}_{\mathfrak{M}};\ \mathfrak{M}_L;\ \mathfrak{M}_{AL};\ \mathfrak{M}_* \equiv (\mathfrak{M}_L)_{\mathfrak{M}};\ \mathfrak{M}_\sharp \equiv (\mathfrak{M}_{AL})_{\mathfrak{M}},$$

*connected with a class* $\mathfrak{M}$.—Having in mind the propositions of §§ 22*a*, 26, we note the following interrelated propositions:

$$\mathfrak{M}^{\mathfrak{N}};\ \mathfrak{N}^{\mathfrak{M}_\sharp}. \tag{1}$$

$$\mathfrak{M}^{D_1} . \supset . \mathfrak{N}^{\text{dominated by } \mathfrak{A}\mathfrak{M}} . \mathfrak{N}^{D_1}. \tag{2}$$

$$\mathfrak{M}^{D_1} . \supset . \mathfrak{M}_* = (\mathfrak{M}_L)_{(\mathfrak{M}_L)} \cdot \mathfrak{M}_\sharp = (\mathfrak{M}_{AL})_{(\mathfrak{M}_{AL})}. \tag{3}$$

$$\mathfrak{M}^{D_1} . \supset . \mathfrak{N}^{D_1} . \mathfrak{M}_*^{LD_1} . \mathfrak{M}_\sharp^{ALD_1}. \tag{4}$$

$$\begin{aligned} \mathfrak{M}^{D} . \supset . \mathfrak{N}^{D} . \mathfrak{M}_*^{LCD} . \mathfrak{M}_* = \mathfrak{M}_{LC} \cdot \mathfrak{M}_{**} = \mathfrak{M}_* \\ . \mathfrak{M}_\sharp^{ALCD} . \mathfrak{M}_\sharp = \mathfrak{M}_{ALC} \cdot \mathfrak{M}_{\sharp\sharp} = \mathfrak{M}_\sharp. \end{aligned} \tag{5}$$

$$\mathfrak{M}^{A} . \supset . \mathfrak{M}_{\mathfrak{M}}^{A}. \tag{6}$$

$$\mathfrak{M}_L^{A} . \supset . \mathfrak{M}_*^{A} . \mathfrak{M}_* = \mathfrak{M}_\sharp. \tag{7}$$

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These types are all of the form (6). The $*$-extension $\mathfrak{M}_*$ of a class $\mathfrak{M}$ of functions contains the sum of every series of functions of $\mathfrak{M}$, which converges for some modified type of convergence of the form (6) uniformly on $\mathfrak{P}$ as to some function of $\mathfrak{M}$, and in fact it consists of all such sums.

(8) $$\mathfrak{M}_*{}^A \,.\supset .\, A(\mathfrak{M}_L)^{\mathfrak{M}*}.$$

(9) $$\mathfrak{M}^D \,.\, \mathfrak{M}_*{}^A \,.\supset .\, \mathfrak{M}_* = \mathfrak{M}_\sharp = \mathfrak{M}_{ALC}.$$

(10) $$(\mathfrak{M}_*)_{\mathfrak{M}} = \mathfrak{M}_* \,.\, A(\mathfrak{M}_L)^{\mathfrak{M}*} \,.\supset .\, \mathfrak{M}_*{}^A.$$

(11) $$\mathfrak{M}_*{}^C \,.\, A(\mathfrak{M}_L)^{\mathfrak{M}*} \,.\supset .\, \mathfrak{M}_*{}^A.$$

(12) $$\mathfrak{M}^D \,.\, A(\mathfrak{M}_L)^{\mathfrak{M}*} \,.\supset .\, \mathfrak{M}_*{}^A \,.\, \mathfrak{M}_* = \mathfrak{M}_\sharp = \mathfrak{M}_{ALC}.$$

THEOREM I. *If the class $\mathfrak{M}$ has the dominance property $D$, the class $\mathfrak{M}_{LC}$, viz., the class $\mathfrak{M}$ extended so as to be linear and closed, is dominated by the class $\mathfrak{A}\mathfrak{M}$ and has the dominance property $D$. $\mathfrak{M}_{LC}$ is given by the formula:*

$$\mathfrak{M}_{LC} = (\mathfrak{M}_L)_{\mathfrak{M}} = \mathfrak{M}_*,$$

*which, in case $\mathfrak{M}^{LC}$ or $\mathfrak{M}^L$, takes the simpler form, $\mathfrak{M}_{LC} = \mathfrak{M}$ or $\mathfrak{M}_{\mathfrak{M}}$ respectively. Furthermore, if $\mathfrak{M}_*$ contains $A(\mathfrak{M}_L)$, in particular, if $\mathfrak{M}_L$ is absolute, then $\mathfrak{M}_*$ is absolute and is the extension $\mathfrak{M}_{ALC}$.*

THEOREM II. *If the class $\mathfrak{M}$ has the dominance property $D$, the class $\mathfrak{M}_{ALC}$, viz., the class $\mathfrak{M}$ extended so as to be absolute, linear, and closed, is dominated by the class $\mathfrak{A}\mathfrak{M}$ and has the dominance property $D$. $\mathfrak{M}_{ALC}$ is given by the formula:*

$$\mathfrak{M}_{ALC} = (\mathfrak{M}_{AL})_{\mathfrak{M}} = \mathfrak{M}_\sharp,$$

*which, in the various cases,*

$$\mathfrak{M}^{ALC};\ \ \mathfrak{M}^{AL};\ \ \mathfrak{M}_A{}^L;\ \ A(\mathfrak{M}_L)^{\mathfrak{M}*},\ \textit{in particular},\ \mathfrak{M}_L{}^A,$$

*assumes the simpler forms:*

$$\mathfrak{M}_{ALC} = \mathfrak{M};\ \ \mathfrak{M}_{\mathfrak{M}};\ \ (\mathfrak{M}_A)_{\mathfrak{M}};\ \ (\mathfrak{M}_L)_{\mathfrak{M}} = \mathfrak{M}_*.$$

### *The homogeneity of the properties $L$, $C$, $D$. § 45.*

45. A property $P$ of one or more classes $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$ is *homogeneous* in case

$$\phi^{\text{nowhere zero}} :\supset: (\mathfrak{M}_1,\ \mathfrak{M}_2,\ \cdots)^P .\sim. (\phi\mathfrak{M}_1,\ \phi\mathfrak{M}_2,\ \cdots)^P,$$

and *positively-homogeneous* in case

$$\phi^{\text{everywhere positive}} :\supset: (\mathfrak{M}_1,\ \mathfrak{M}_2,\ \cdots)^P .\sim. (\phi\mathfrak{M}_1,\ \phi\mathfrak{M}_2,\ \cdots)^P;$$

viz., in case the property $P$ is invariant under multiplication of its various argument classes $\mathfrak{M}$ of functions by any (one and the same) function $\phi$ on $\mathfrak{P}$ which for $\substack{\text{no } p \text{ vanishes} \\ \text{every } p \text{ is positive}}$.

*Unipartite properties* $P$ of classes $\mathfrak{M}$ are properties of classes $\mathfrak{M}$ taken singly. *Multipartite properties* $P$ of classes $\mathfrak{M}$ are properties of classes $\mathfrak{M}$ taken in systems,[*] usually of a specified type; they are thus unipartite properties of such systems of classes.

Evidently *the properties:*

$$L\,;\ C\,;\ D\,;\ D_1\,;\ D_2,$$

*are homogeneous unipartite properties* of classes $\mathfrak{M}$, while *the properties:*

$$D_0\,;\ A,$$

*are positively-homogeneous unipartite properties.*

Amongst the homogeneous properties $P$ are moreover systems of linear homogeneous relations with constant coefficients amongst functions (not amongst functional values), e. g., linear homogeneous transformations of $n$-partite functions on $\mathfrak{P}$ to $\mathfrak{A}$, — the classes $\mathfrak{M}$ being in this case singular: consisting each of a single function.

For multipartite properties $P$, in the homogeneity just defined the various classes $\mathfrak{M}_1, \mathfrak{M}_2, \cdots$ having collectively the property or relation $P$ enter as of like weight or dimension; the homogeneity is *isobaric*. By the introduction of discriminating weights more general types of homogeneity of multipartite properties may be defined.

### *The complete independence of the properties:* $L;\ C;\ D;\ D_0$ *or* $A$.
### *The complete existential theory of the properties:*
### $L;\ C;\ D;\ D_0;\ A.$ §§ 46–18.

46. With respect to systems:

$$\Sigma = (\mathfrak{A}\,;\ \mathfrak{P}\,;\ \mathfrak{M}),$$

we have been considering in particular the properties:

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[*] By the use of the numeral suffixes, 1, 2, $\cdots$ in the system $(\mathfrak{M}_1, \mathfrak{M}_2, \cdots)$ employed in the definition no implication of denumerability of the system of classes is intended.

$$L;\ C;\ D;\ D_0;\ A,$$

of classes $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$. We know that $\mathfrak{M}^A$ implies $\mathfrak{M}^{D_0}$. It is interesting to note that no other general relation holds amongst the five properties, and that, in a sense to be defined directly, the four properties: $L$; $C$; $D$; $D_0$ (or $A$) are completely independent.

47. Consider in general $m$ properties:

$$(1) \qquad P_1;\ P_2;\ \cdots;\ P_m,$$

of systems $\sum$ of a certain type. A system $\sum$ has with respect to the $m$ properties (1) a definite one of the $2^m$ $m$-partite characters:

$$(2) \qquad \begin{array}{c}(++\cdots++);\ (++\cdots+-);\ \cdots\cdots;\\ (--\cdots-+);\ (--\cdots--),\end{array}$$

the part $i$ of the character of the system $\sum$ being $+$ or $-$ according as the system $\sum$ has or does not have the property $P_i$.

*The complete (elementary) existential theory* of the $m$ properties (1) of systems $\sum$ is the body of $2^m$ propositions stating for the various characters (2) that there exists or that there does not exist a system $\sum$ (of the type in question) having the character in question.

The $m$ properties (1) are *completely independent (and mutually consistent)* in case the $2^m$ propositions of the complete theory are propositions of existence.

The $m$ properties are independent in the usual sense in case there are $m$ propositions of existence, viz., in case there exist $m$ systems $\sum$ failing to have each precisely one of the $m$ properties, so that no one of the $m$ properties is implied by the remaining $m-1$ properties. Sets of independent properties (postulates) fundamental for various mathematical disciplines have recently been exhibited. There would be some interest in the existential theories of those sets and in the determination (if possible) of sets of completely independent fundamental properties.

48. THEOREM. *For the five properties:*

(1) $$L;\ C;\ D;\ D_0;\ A,$$

*of systems:*

(2) $$\Sigma = (\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M}),$$

*viz., of classes* $\mathfrak{M}$ *of functions on* $\mathfrak{P}$ *to* $\mathfrak{A}$, *the complete existential theory consists of* $2^3 = 8$ *propositions of non-existence, viz., that no system*

(3) $$\Sigma^{\cdots - A}$$

*exists, and of* $2^5 - 2^3 = 24$ *propositions of existence, so that, in particular, the properties:*

(4) $$L;\ C;\ D;\ D_0,$$

*and likewise the properties:*

(5) $$L;\ C;\ D;\ A,$$

*are completely independent.*

The non-existences (3) are expressed by the proposition:

(6) $$\mathfrak{P} \,.\, \mathfrak{M}^{A} \,.\supset.\, \mathfrak{M}^{D_0},$$

already referred to. We notice further that

(7) $$\mathfrak{P}^{\text{finite}} \,.\, \mathfrak{M}^{L} \,.\supset.\, \mathfrak{M}^{CD};$$

(8) $$\mathfrak{P}^{\text{dual}} \,.\, \mathfrak{M}^{LD_0} \,.\supset.\, \mathfrak{M}^{A};$$

(9) $$\mathfrak{P}^{\text{singular}} \,.\, \mathfrak{M} \,.\supset.\, \mathfrak{M}^{D};$$

(10) $$\mathfrak{P}^{\text{singular}} \,.\, \mathfrak{M}^{L} \,.\supset.\, \mathfrak{M}^{A}.$$

The effect of these propositions is to cut down the 24 characters existing for $\mathfrak{P}$ unrestricted to 15 for $\mathfrak{P}$ finite; to 14 for $\mathfrak{P}$ dual; to 7 for $\mathfrak{P}$ singular.

A class $\mathfrak{M}$ of functions on $\mathfrak{P}^{\text{finite}}$ to $\mathfrak{A}$ is a point-set in real flat space of say $n$ dimensions. A linear class $\mathfrak{M}$ is the locus of points whose $n$ coördinates satisfy a system of linear homogeneous equations, whose coefficients all vanish in case the point-set is the space itself: $\mathfrak{M} \equiv \mathfrak{M}^{\Pi_n}$.

Examples of classes $\mathfrak{M}$ of the various characters will be given as follows:

$$(7;\ 7;\ 1) \quad \text{on} \quad \mathfrak{P}^{\mathrm{II}_n} \quad (n = 1;\ 2;\ 3)$$

and

$$9 \text{ on } \mathfrak{P}^{\text{denumerably infinite}} = \mathfrak{P}^{\mathrm{III}}.$$

These examples are sufficient, since the character, as to the properties (1), of a class $\mathfrak{M}$ on $\mathfrak{P}$ is invariant under the enlargement of $\mathfrak{P}$ to a class $\mathfrak{P}'$ by the adjunction of a class $\mathfrak{P}''$ having no elements in common with $\mathfrak{P}$, the functions $\mu$ of $\mathfrak{M}$ being extended to functions $\mu'$ of $\mathfrak{M}'$ vanishing on $\mathfrak{P}''$.

We observe that there is no loss of existing characters in passing from $\mathfrak{P}$ unrestricted to $\mathfrak{P}$ denumerable nor in passing from $\mathfrak{P}$ finite to $\mathfrak{P}$ of three elements, and that the 15 characters on $\mathfrak{P}$ of three elements include all the 12 existing non-linear characters.

48*a*. *Examples for the seven characters for $\mathfrak{P}$ singular.*—

| *Character.* | *Functions of the class $\mathfrak{M}$.* |
|---|---|
| $(+ + + + +)$ | $0$ ($\mathfrak{M} = \mathfrak{M}^{\mathrm{I}} = \mathfrak{A}$ is the only other linear $\mathfrak{M}$.) |
| $(- + + + +)$ | $1$ |
| $(- + + + -)$ | $-1,\ 2$ |
| $(- + + - -)$ | $-1$ |
| $(- - + + +)$ | $1/n \quad (n)$* |
| $(- - + + -)$ | $-1/n,\ 2/n \quad (n)$ |
| $(- - + - -)$ | $-1/n \quad (n)$ |

48*b*. *Examples for the seven additional characters for $\mathfrak{P}$ dual.*—

| *Character.* | *Functions of the class $\mathfrak{M}$.* |
|---|---|
| $(+ + + - -)$ | $(a, -a) \quad (a)$* |
| $(- + - + +)$ | $(1, 0),\ (0, 1)$ |
| $(- + - + -)$ | $(-1, 0),\ (2, 0),\ (0, 1)$ |
| $(- + - - -)$ | $(-1, 0),\ (0, 1)$ |
| $(- - - + +)$ | $(1/n, 0),\ (0, 1) \quad (n)$ |
| $(- - - + -)$ | $(-1/n, 0),\ (0, 1),\ (2/n, 0) \quad (n)$ |
| $(- - - - -)$ | $(-1/n, 0),\ (0, 1) \quad (n)$ |

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* The symbols $(n)$, $(a)$ denote *for every n*, *for every a*.

*Example for the one additional character for $\mathfrak{P}$ of three* . —

| *Character.* | *Functions of the class $\mathfrak{M}$.* |
|---|---|
| $(++++-)$ | $(a_1, a_2, a_1 + a_2)$ $(a_1 a_2)$ |

*Examples for the nine additional characters for $\mathfrak{P}$ de- bly infinite.* —

ne $2^4 = 16$ linear characters the relation 6 of § 48 cuts of every four. The remaining 12 linear characters he three *LCD* for $\mathfrak{P}$ finite, viz., $(+++++)$ for $\mathfrak{P}$ r, $(++++-)$ for $\mathfrak{P}$ of three elements, $(+++--)$ lual, and the remaining nine for $\mathfrak{P}$ denumerably infinite.

$$\mathfrak{M}' \equiv [\mu']; \quad \mathfrak{M}'' \equiv [\mu'']; \quad \mathfrak{M}''' \equiv [\mu''']$$

respective classes of all functions :

$\mathfrak{P}' \equiv [p' = 1, 2, 3, \cdots]$ *ultimately zero* ;*

$\mathfrak{P}'' \equiv [p'' = 1, 2, 3, \cdots]$ *ultimately constant* ;†

*n $\mathfrak{P}'\mathfrak{P}''$ for every $p''$ of class $\mathfrak{M}'$ on $\mathfrak{P}'$ and for every $p'$ of lass $\mathfrak{M}''$ on $\mathfrak{P}''$.*

se classes have the characters :

$+-++)$; $\mathfrak{M}''$: $(+-+++)$; $\mathfrak{M}'''$: $(+--++)$,

ve have above listed classes having the characters :

$[\mu^3]$: $(++++-)$; $\mathfrak{M}^2 \equiv [\mu^2]$: $(+++--)$,

$\equiv [j = 1, 2, 3]$; $\mathfrak{P}^2 \equiv [i = 1, 2]$ respectively. The nine $\mathfrak{M}$ to be exhibited are the three classes $\mathfrak{M}'$, $\mathfrak{M}''$, $\mathfrak{M}'''$ and classes arising from them by composition with $\mathfrak{M}^3$ and

| *Character.* | *Class $\mathfrak{M}$.* |
|---|---|
| $(++-++)$ | $\mathfrak{M}'$ on $\mathfrak{P}'$ |
| $(++-+-)$ | $(\mathfrak{M}^3\mathfrak{M}')_L$ on $\mathfrak{P}^3\mathfrak{P}'$ |

---

t is, such that there exists $p_0'$ (dependent on $\mu'$) for which $p' \geqq p_0'$ $'_{p'} = 0$.

is, such that there exists $p_0''$ (dependent on $\mu''$) for which $p'' \geqq p_0''$ $''_{p''} = \mu''_{p_0''}$.

| *Character.* | *Class* $\mathfrak{M}$. |
|---|---|
| $(++---)$ | $(\mathfrak{M}^2\mathfrak{M}')_L$ on $\mathfrak{P}^2\mathfrak{P}'$ |
| $(+-+++)$ | $\mathfrak{M}''$ on $\mathfrak{P}''$ |
| $(+-++-)$ | $(\mathfrak{M}^3\mathfrak{M}'')_L$ on $\mathfrak{P}^3\mathfrak{P}''$ |
| $(+-+--)$ | $(\mathfrak{M}^2\mathfrak{M}'')_L$ on $\mathfrak{P}^2\mathfrak{P}''$ |
| $(+--++)$ | $\mathfrak{M}'''$ on $\mathfrak{P}'\mathfrak{P}''$ |
| $(+--+-)$ | $(\mathfrak{M}^3\mathfrak{M}''')_L$ on $\mathfrak{P}^3\mathfrak{P}'\mathfrak{P}''$ |
| $(+----)$ | $(\mathfrak{M}^2\mathfrak{M}''')^L$ on $\mathfrak{P}^2\mathfrak{P}'\mathfrak{P}''$ |

Here the classes : $\mathfrak{P}'$ ; $\mathfrak{P}^3\mathfrak{P}'$; $\cdots$ ; $\mathfrak{P}^2\mathfrak{P}'\mathfrak{P}''$, are obviously denumerable. Further, the notations: $(\mathfrak{M}^3\mathfrak{M}')_L$, etc., denote classes of functions on $\mathfrak{P}^3\mathfrak{P}'$, etc.; viz., e. g., $\mathfrak{M}^3\mathfrak{M}'$ is the product class consisting of all products $\mu^3\mu'$ and $(\mathfrak{M}^3\mathfrak{M}')_L$ is the class $\mathfrak{M}^3\mathfrak{M}'$ extended so as to be linear.

### *Functional characterization of classes* $\mathfrak{P}$ *of elements.* § 49.

49. From the complete existential theory of properties of classes $\mathfrak{M}$ of functions on $\mathfrak{P}$ to $\mathfrak{A}$ we are led to characterize certain classes $\mathfrak{P}$ of elements functionally, by means of properties of their classes $\mathfrak{M}$ of functions. Thus, using the properties $L$, $C$, $D$, $D_0$, $A$ positively, in the following theorem we characterize the classes (I) $\mathfrak{P}$ singular, ($\mathrm{II}_{12}$) $\mathfrak{P}$ singular or dual, (II) $\mathfrak{P}$ finite.

THEOREM. *In the following three sets* (I, $\mathrm{II}_{12}$, II) *of properties of a class* $\mathfrak{P}$ *of elements any two properties of the same set are equivalent.*

(I 1) singular.
(I 2) $\mathfrak{M} \,.\supset.\, \mathfrak{M}^{D}$.
(I 2′) $\mathfrak{M}^{CA} \,.\supset.\, \mathfrak{M}^{D}$.
(I 3) $\mathfrak{M}^{L} \,.\supset.\, \mathfrak{M}^{CDD_0A}$.
(I 3′) $\mathfrak{M}^{LCD} \,.\supset.\, \mathfrak{M}^{D_0}$.

($\mathrm{II}_{12}$ 1) singular or dual.
($\mathrm{II}_{12}$ 2) $\mathfrak{M}^{LD_0} \,.\supset.\, \mathfrak{M}^{CDA}$.
($\mathrm{II}_{12}$ 2′) $\mathfrak{M}^{LCDD_0} \,.\supset.\, \mathfrak{M}^{A}$.

(II 1) finite.

(II 2) $\mathfrak{M}^{L} \,.\supset .\, \mathfrak{M}^{CD}.$

(II 2′) $\mathfrak{M}^{LD\Delta} \,.\supset .\, \mathfrak{M}^{C}.$

(II 2″) $\mathfrak{M}^{LC\Delta} \,.\supset .\, \mathfrak{M}^{D}.$

(II 3) $(\mathfrak{M}^{L} \ni p \,.\supset .\, \exists\ \mu \ni \mu_p \neq 0) : \supset : \exists\ \mu \ni p \,.\supset .\, \mu_p \neq 0.$

(II 4) $(\mathfrak{M} \ni \exists\ \mu \ni p \,.\supset .\, \mu_p \neq 0) \,.\supset .\, \mathfrak{M}^{D}.$

(II 4′) $(\mathfrak{M}^{L} \ni \exists\ \mu \ni p \,.\supset .\, \mu_p \neq 0) \,.\supset .\, \mathfrak{M}^{D}.$

(II 5) $\underset{n}{\mathrm{L}}\lrcorner\,\mu_n = \theta \,.\supset .\, \underset{n}{\mathrm{L}}\lrcorner\,\mu_n = \theta \quad (\mathfrak{P};\ 1)$

For a finite $\mathfrak{P}$ the conditions (II 3, 4, 4′, 5) obviously hold. That they do not hold for $\mathfrak{P}^{III}$ appears from the examples:

ad II 3 : $\mathfrak{M}$ is the class $\mathfrak{M}'$ of § 48*d*;

ad II 4, 4′: $\mathfrak{M}$ is the class of all polynomials in $p$ with real coefficients;

ad II 5 : $\mu_{np} = \dfrac{p}{n}\ (np);\ \theta_p = 0\ (p).$

49*a*. *Propositions.*—Denoting for a class $\mathfrak{P}$ by $\mathfrak{M}^{\mathrm{all}}$ the class of all functions on $\mathfrak{P}$ to $\mathfrak{A}$, for a *denumerable* class $\mathfrak{P}$ we have the two propositions:

(1) $\mathfrak{M}^{\mathrm{all}} \,.\supset .\, \mathfrak{M}^{D}.$

(2) $\mathfrak{M}^{\mathrm{all}} : \supset : \underset{n}{\mathrm{L}}\lrcorner\,\mu_n = \mu \,.\supset .\, \underset{n}{\mathrm{L}}\lrcorner\,\mu_n = \mu \quad (\mathfrak{P};\ \mathfrak{M}),$

of which the former has been considered in § 23*c*5. The question arises whether denumerable classes $\mathfrak{P}$ are functionally characterized by the validity of one or both of these propositions.

We proceed to prove the second proposition: *for a denumerable class $\mathfrak{P}$, an assumed convergence:*

(3) $\underset{n}{\mathrm{L}}\lrcorner\,\mu_n = \mu,$

*is uniform as to some scale function* $\sigma$. In terms of an auxiliary sequence $\{e_m\}$:

($4_1$, $4_2$) $0 \lessdot e_{m+1} \lessdot e_m \quad (m); \qquad \underset{m}{\mathrm{L}}\lrcorner\,e_m = 0,$

of positive numbers, e. g., $e_m = 1/m \ (m)$, in view of (3) there exists a sequence $\{n_m\}$ of positive integers satisfying the conditions:

$$n_{m+1} > n_m \quad (m); \tag{$5_1$}$$

$$p \leqq m \,.\, n \geqq n_m \,.\supset.\, A(\mu_{np} - \mu_p) \leqq e_m, \tag{$5_2$}$$

and, accordingly there exists a function $\sigma$ satisfying the conditions:

$$\sigma_p \geqq 1 \quad (p); \tag{$6_1$}$$

$$p \,.\, n \leqq n_p \,.\supset.\, A(\mu_{np} - \mu_p) \leqq e_p \sigma_p \,. \tag{$6_2$}$$

Any such function $\sigma$ is effective as such a scale function $\sigma$:

$$\mathop{\mathrm{L}}_{n} \mu_n = \mu \quad (\mathfrak{P}\,;\, \sigma). \tag{7}$$

The proof of (7) runs through the relations:

$$p \leqq m \,.\, n \geqq n_m \,.\supset.\, A(\mu_{np} - \mu_p) \leqq e_m \sigma_p \qquad (5_2,\, 6_1); \tag{8}$$

$$p \,.\, n \geqq n_p \,.\supset.\, A(\mu_{np} - \mu_p) \leqq e_p \sigma_p \qquad (8); \tag{9}$$

$$p \,.\, n \,.\supset.\, A(\mu_{np} - \mu_p) \leqq e_p \sigma_p \qquad (6_2,\, 9); \tag{10}$$

$$m \leqq p \,.\, n \,.\supset.\, A(\mu_{np} - \mu_p) \leqq e_m \sigma_p \qquad (10,\, 4_1); \tag{11}$$

$$p \,.\, n \geqq n_m \,.\supset.\, A(\mu_{np} - \mu_p) \leqq e_m \sigma_p \qquad (8,\, 11), \tag{12}$$

where (8) is implied by $(5_2,\, 6_1)$, etc., (12) in view of $(4_2)$ implying (7).

In Part I we have studied certain interrelations of systems:

$$(\mathfrak{A}\,;\, \mathfrak{P}\,;\, \mathfrak{M}),$$

viz., classes $\mathfrak{M}$ of real- and single-valued functions, having a common range $\mathfrak{P}$,—in particular with respect to certain closure and dominance properties.

# PART II.

## FUNDAMENTAL COMPOSITION PROPERTIES OF CLASSES OF FUNCTIONS OF A GENERAL VARIABLE.

### *Introduction.* § 50.

50. Part II is devoted to the study of certain compositional interrelations of systems:

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M}),$$

on independent ranges $\mathfrak{P}$.

This study falls into four sections. In the first section (§§ 51–55) we define the composition of classes $\mathfrak{P}$ of elements and of classes $\mathfrak{M}$ of functions and show that the two genera of systems $(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M})$ with the properties $D$; $LCD$ are each closed under $*$-composition; here of course the first genus contains the second which contains in particular the systems I–IV; and every $*$-composite of two systems of the first genus is a system of the second genus.

In the second section (§§ 56–67) we raise the question of the functional characterization of the $*$-composite of two systems $(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M})$ and obtain characterizations of the $*$-composites of the systems I–IV with arbitrary systems of the genus $LCD$. These characterizations are of a uniform type $K_{12*}$ (cf. § 67) depending upon certain developments $\Delta$ of the classes $\mathfrak{P}$ of elements in the cases I–IV.

In the fourth section (§§ 75–84) in terms of the general development $\Delta$ of the general class $\mathfrak{P}$ of elements we define a genus of systems:

$$(\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M}),$$

whose $*$-composites with systems of the genus $LCD$ have functional characterizations of type $K_{12*}$. This genus contains the systems I–IV and is closed under $*$-composition. The characterizations of type $K_{12*}$ are defined in terms of certain relations:[*] $K_1$; $K_2$, which are in turn defined in terms of the development $\Delta$.

In the third section (§§ 68–74) we introduce general relations: $K_1$; $K_2$, and show that the genus of systems:

$$(\mathfrak{A};\ \mathfrak{P};\ K_1,\ K_2;\ \mathfrak{M}),$$

with dominance property $D$ whose $*$-composites with systems of the genus $LCD$ have functional characterizations of type $K_{12*}$ is closed under $*$-composition.

The theorems of closure of certain suitably conditioned genera of systems:

$$(\mathfrak{A};\ \mathfrak{P};\ \mathfrak{M});\quad (\mathfrak{A};\ \mathfrak{P};\ K_1,\ K_2;\ \mathfrak{M});\quad (\mathfrak{A};\ \mathfrak{P};\ \Delta;\ \mathfrak{M}),$$

under composition of classes $\mathfrak{P}$; relations $K_1$, $K_2$; developments $\Delta$; and $*$-composition of classes $\mathfrak{M}$, occur in §§ 55; 74; 84. These theorems indicate the scope of the notions under investigation. We note that the dominance property $D$ enters as a condition on the classes $\mathfrak{M}$ and that relative uniformity of convergence enters in the definition of the closure and of the $*$-extension and the $*$-composition of classes $\mathfrak{M}$.

### *Composition and reduction of classes of elements.* §§ 51–52.

51. *Composition.* — From two classes

$$\mathfrak{P}' \equiv [p'], \qquad \mathfrak{P}'' \equiv [p'']$$

of elements we derive the *product class*

$$\mathfrak{P}'\mathfrak{P}'' \equiv [(p', p'')] \equiv [p'p''],$$

i. e., the class $\mathfrak{P} \equiv [p]$ whose elements $p \equiv (p', p'')$ are bipartite, the first part $p'$ ranging over $\mathfrak{P}'$ and the second part $p''$ ranging independently over $\mathfrak{P}''$. In practice and with occasional caution we replace the notation $(p', p'')$ by $p'p''$.

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[*] These relations $K_1$, $K_2$ are properties $K_1$, $K_2$ on $\mathfrak{P}\mathfrak{Z}$, $\mathfrak{P}\mathfrak{P}\mathfrak{Z}$, where $\mathfrak{Z}$ denotes the class $[m]$ of positive integers, in terms of which are defined in § 72 the bipartite properties $K_1$, $K_2$ of functions and classes of functions on $\mathfrak{P}$, and their associated properties.

Similarly we obtain from a finite number of classes $\mathfrak{P}', \mathfrak{P}'', \cdots, \mathfrak{P}^{(n)}$ the product class $\mathfrak{P}'\mathfrak{P}'' \cdots \mathfrak{P}^{(n)} \equiv [p'p'' \cdots p^{(n)}]$.

For present purposes this composition of classes of elements is associative.

The classes $\mathfrak{P}', \mathfrak{P}''$ may be the same. We write

$$\mathfrak{P}\mathfrak{P} \equiv [(p_1, p_2)] \equiv [p_1 p_2],$$

and similarly, for example,

$$\mathfrak{P}\mathfrak{P}'\mathfrak{P}\mathfrak{P}''\mathfrak{P}' \equiv [(p_1, p_1', p_2, p'', p_2')] \equiv [p_1 p_1' p_2 p'' p_2'].$$

Thus $p'$, $p''$ are generic elements of the respective classes $\mathfrak{P}', \mathfrak{P}''$ conceptually distinct but not necessarily actually distinct, while $p_1$, $p_2$ are independent (conceptually distinct) generic elements of the class $\mathfrak{P}$.

51*a*. *Remark.*—It is to be noticed at once that the theory of functions and classes of functions on $\mathfrak{P}$ to $\mathfrak{A}$ as developed in Part I is applicable to functions and classes of functions on $\mathfrak{P}'\mathfrak{P}''$, etc., to $\mathfrak{A}$, and this is to be thought of in the sense that $\mathfrak{P}$ in its generality includes $\mathfrak{P}'\mathfrak{P}''$, etc.

52. *Reduction.*—$\mathfrak{P}$ being a class of elements, a *reduction* $R$ is the transformation of $\mathfrak{P}$ into a subclass of itself, in notation $\mathfrak{P}_R \equiv [p_R]$, *the class* $\mathfrak{P}$ *reduced by the reduction* $R$. We may speak of the *reducing condition* $R$, the class $\mathfrak{P}_R$ being the class of all elements $p$ of $\mathfrak{P}$ satisfying the condition $R$:

$$\mathfrak{P} \equiv [p]; \quad \mathfrak{P}_R \equiv [p_R] \equiv [\text{all } p^R].$$

A reduction $R$ transforms $\mathfrak{P}$ into the *reduced class* $\mathfrak{P}_R$ and a function $F$ on $\mathfrak{P}$ to a class $\mathfrak{P}'$ (cf. § 4) into the *reduced function* $F_R$ on $\mathfrak{P}_R$ to $\mathfrak{P}'$:

$$F \equiv (F_p \mid p^{\mathfrak{P}}); \quad F_R \equiv (F_p \mid p^{\mathfrak{P}R}),$$

and a class $\mathfrak{F}$ of functions on $\mathfrak{P}$ to $\mathfrak{P}'$ into the *reduced class* $\mathfrak{F}_R$ of functions on $\mathfrak{P}_R$ to $\mathfrak{P}'$:

$$\mathfrak{F} \equiv [F]; \quad \mathfrak{F}_R \equiv [F_R].$$

The reduction $R$ is said to be *applicable* to the class $\mathfrak{P}$ and accordingly to functions and classes of functions on $\mathfrak{P}$.

The reduction $R$ is the *identity reduction*, in case the reduced class $\mathfrak{P}_R$ is the class $\mathfrak{P}$ itself.

52*a*. *Propositions.* — $R$ being a reduction applicable to $\mathfrak{P}$ and $\mathfrak{M}$ and $\mathfrak{S}$ being classes of functions on $\mathfrak{P}$ to $\mathfrak{A}$, we have the propositions:

(1) $$(A\mathfrak{M})_R = A(\mathfrak{M}_R).$$

(2) $$(\mathfrak{M}_A)_R = (\mathfrak{M}_R)_A .$$

(3) $$(\mathfrak{M}_L)_R = (\mathfrak{M}_R)_L .$$

(4) $$\mathfrak{S}^{\mathfrak{M}} . \supset . \mathfrak{S}_R^{\mathfrak{M}_R}.$$

(5) $$\mathfrak{M}^A . \supset . \mathfrak{M}_R^A.$$

(6) $$\mathfrak{M}^L . \supset . \mathfrak{M}_R^L.$$

(7) $$\mathfrak{M}^{D_1} . \supset . \mathfrak{M}_R^{D_1}.$$

(8) $$\mathfrak{M}^D . \supset . \mathfrak{M}_R^D.$$

(9) $$(\mathfrak{M}_{\mathfrak{S}})_R \text{ is contained in } (\mathfrak{M}_R)_{(\mathfrak{S}_R)} .$$

(10) $$(\mathfrak{M}_*)_R \text{ is contained in } (\mathfrak{M}_R)_* .$$

(11) $$\mathfrak{M}_*^A . \supset . A\big((\mathfrak{M}_R)_L\big)^{(\mathfrak{M}_R)_*}.$$

(12) $$\mathfrak{M}_*^A . \mathfrak{M}_R^D . \supset . (\mathfrak{M}_R)_*^A . (\mathfrak{M}_R)_* = (\mathfrak{M}_R)_\# .$$

Here (11) is a corollary and a generalization of § 44*a*8, in view of (1, 3, 4, 9), and (12) is a corollary of (11) in view of § 44*a*12.

52*b*. A unipartite property $P$ of classes of functions is *invariant under reduction of classes* in case

$A_1$) $$\mathfrak{M}^P . R . \supset . \mathfrak{M}_R^P,$$

where $R$ denotes a reducing substitution applicable to $\mathfrak{P}$.

Similarly, a bipartite property $P$ is invariant in case

$A_2$) $$(\mathfrak{M}, \mathfrak{S})^P . R . \supset . (\mathfrak{M}_R, \mathfrak{S}_R)^P.$$

Thus the properties:

(1) $$P = A;\ D_1;\ D,$$

are unipartite invariant properties, and the properties:

(2) $P =$ *contains; dominates; is contained in the L-extension of,* are bipartite invariant properties.

### *Composition of classes of functions.* §§ 53–55.

53. *Multiplication.*—From two classes:

$$\mathfrak{M}' \equiv [\mu']; \qquad \mathfrak{M}'' \equiv [\mu''],$$

of functions:

$$\mu' \equiv (\mu'_{p'} \,|\, p'); \qquad \mu'' \equiv (\mu''_{p''} \,|\, p''),$$

on the classes $\mathfrak{P}'$, $\mathfrak{P}''$ respectively to $\mathfrak{A}$, arises by multiplication of constituent functions the *product class:*

$$\mathfrak{M}'\mathfrak{M}'' \equiv [\mu'\mu''],$$

of functions:

$$\mu'\mu'' \equiv (\mu'_{p'}\mu''_{p''} \,|\, p'p''),$$

on the product class $\mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$.

The similarity of notations and the disparity of notions for the two product classes, of elements and of functions, in practice with occasional caution cause no confusion.

This process of multiplication of two classes of functions, as extended to any finite number of classes, is obviously associative and commutative, it being understood that in the permutations of the constituent classes each class carries with it its own variable.

53*a*. Suppose the two classes $\mathfrak{P}'$, $\mathfrak{P}''$ are the same:

$$\mathfrak{P}' \equiv \mathfrak{P}'' \equiv \mathfrak{P} \equiv [\, p \,]; \qquad \mathfrak{P}'\mathfrak{P}'' \equiv \mathfrak{P}\mathfrak{P} \equiv [\, p_1 p_2 \,].$$

Then a function $\mu$ on $\mathfrak{P}$:

$$\mu \equiv (\mu_p \,|\, p),$$

may be considered as a function $\mu$ on $(\mathfrak{P}\mathfrak{P})_R$:

$$\mu \equiv (\mu_{pp} \,|\, pp\,) \qquad\qquad (\mu_{pp} = \mu_p\ (p)),$$

where $(\mathfrak{P}\mathfrak{P})_R$ is $\mathfrak{P}\mathfrak{P}$ reduced by the reducing substitution:

$$R: \quad p_1 = p_2 = p.$$

Conversely, $R$ reduces a function $\mu$ on $\mathfrak{P}\mathfrak{P}$ to a reduced function $\mu_R$ on $(\mathfrak{P}\mathfrak{P})_R$ which may be considered as a function on $\mathfrak{P}$.

Accordingly, the product $\mathfrak{M}'\mathfrak{M}''$ in the sense of § 14 of two classes of functions on $\mathfrak{P}$ may be considered as the reduced product $(\mathfrak{M}'\mathfrak{M}'')_R$ of the product $\mathfrak{M}'\mathfrak{M}''$ of the two classes in the sense of § 53. In the sequel, in the absence of specification to the contrary, *the notations* $\mathfrak{M}'\mathfrak{M}''$, *etc., are to be understood in the sense of* § 53.

54. *Composition.* — Of the classes of functions arising by composition of two classes $\mathfrak{M}'$, $\mathfrak{M}''$ of functions on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively we consider the following four:

*a*) the product: $\mathfrak{M}'\mathfrak{M}''$ on $\mathfrak{P}'\mathfrak{P}''$;
*b*) the general product: $(\mathfrak{M}'\mathfrak{M}'')_R$ on $(\mathfrak{P}'\mathfrak{P}'')_R$;
*c*) the $*$-composite: $(\mathfrak{M}'\mathfrak{M}'')_*$ on $\mathfrak{P}'\mathfrak{P}''$;
*d*) the general $*$-composite: $(\mathfrak{M}'\mathfrak{M}'')_{R*}$ on $(\mathfrak{P}'\mathfrak{P}'')_R$.

Here $R$ denotes the general reduction applicable to the product $\mathfrak{P}'\mathfrak{P}''$, while the $*$-composite (simple or general) is the $*$-extension in the sense of § 44 of the product (simple or general).

We note by § 52$a$10 that $(\mathfrak{M}'\mathfrak{M}'')_{*R}$ is contained in $(\mathfrak{M}'\mathfrak{M}'')_{R*}$.

In case the reduction $R$ is the identity reduction, the general product is the simple product and the general $*$-composite is the simple $*$-composite.

54$a$. *Propositions.* — A unipartite property $P$ of classes of functions is *invariant under multiplication of classes* in case

$$A_1) \qquad \mathfrak{M}'^{P} \,.\, \mathfrak{M}''^{P} \,.\supset.\, (\mathfrak{M}'\mathfrak{M}'')^{P}.$$

A bipartite property $P$ is invariant in case

$$A_2) \qquad (\mathfrak{M}', \mathfrak{S}')^{P} \,.\, (\mathfrak{M}'', \mathfrak{S}'')^{P} \,.\supset.\, (\mathfrak{M}'\mathfrak{M}'', \mathfrak{S}'\mathfrak{S}'')^{P}.$$

Thus the properties:

$$(1) \qquad P = A;\ D_1;\ D,$$

are unipartite invariant properties, and the properties:

(2) $P =$ *contains; dominates; is contained in the $L$-extension of,*

are bipartite invariant properties.

Concerning classes $\mathfrak{M}'$, $\mathfrak{S}'$, $\mathfrak{M}''$, $\mathfrak{S}''$ of functions and a reducing substitution $R$ applicable to $\mathfrak{P}'\mathfrak{P}''$ we note the propositions:

$$(3) \qquad (\mathfrak{M}'_L\mathfrak{M}''_L)_{RL} = (\mathfrak{M}'\mathfrak{M}'')_{RL}.$$

$$(4) \qquad (\mathfrak{M}'\mathfrak{M}'')_{RL} \text{ contains } (\mathfrak{M}'_L\mathfrak{M}''_L)_R.$$

$$(5) \qquad (\mathfrak{M}'\mathfrak{M}'')_{R*} \text{ contains } (\mathfrak{M}'_L\mathfrak{M}''_L)_{R_{(\mathfrak{M}'\mathfrak{M}'')_R}}.$$

(6) $(\mathfrak{M}_*{}'\mathfrak{M}_*{}'')_{R*}$ contains $(\mathfrak{M}'\mathfrak{M}'')_{R*}$.

(7) $\dagger' . \dagger'' . \mathfrak{S}'^{D_1} . \mathfrak{S}''^{D_1} . \supset . (\mathfrak{M}'\mathfrak{M}'')_{R(\mathfrak{S}'\mathfrak{S}'')_R}$ contains $(\mathfrak{M}'_{\mathfrak{S}'}\mathfrak{M}''_{\mathfrak{S}''})_R$,

where the conditions $\dagger'$, $\dagger''$ are that $\mathfrak{M}'_{\mathfrak{S}'}$, $\mathfrak{M}''_{\mathfrak{S}''}$ are dominated by $\mathfrak{A}\mathfrak{S}'$, $\mathfrak{A}\mathfrak{S}''$ respectively.

(8) $\mathfrak{M}'^{D_1} . \mathfrak{M}''^{D_1} . \supset . (\mathfrak{M}'_L\mathfrak{M}''_L)_{R(\mathfrak{M}'\mathfrak{M}'')_R}$ contains $\big((\mathfrak{M}'_L)_{\mathfrak{M}'}(\mathfrak{M}''_L)_{\mathfrak{M}''}\big)_R = (\mathfrak{M}'_*\mathfrak{M}''_*)_R$.

(9) $\mathfrak{M}'^{D_1} . \mathfrak{M}''^{D_1} . \supset . (\mathfrak{M}'\mathfrak{M}'')_{R*}$ contains $(\mathfrak{M}'_*\mathfrak{M}''_*)_R$.

(10) $\mathfrak{M}'^{D} . \mathfrak{M}''^{D} . \supset . (\mathfrak{M}'\mathfrak{M}'')_R{}^D . (\mathfrak{M}'\mathfrak{M}'')_{R**} = (\mathfrak{M}'\mathfrak{M}'')_{R*}$.

(11) $\mathfrak{M}'^{D} . \mathfrak{M}''^{D} . \supset . (\mathfrak{M}'\mathfrak{M}'')_{R*}$ contains $(\mathfrak{M}'_*\mathfrak{M}''_*)_{R*}$.

(12) $\mathfrak{M}'^{D} . \mathfrak{M}''^{D} . \supset . (\mathfrak{M}'\mathfrak{M}'')_{R*} = (\mathfrak{M}'_*\mathfrak{M}'_*)_{R*} = (\mathfrak{M}'_*\mathfrak{M}''_*)_{R*}$.

Here (7) is a corollary of § 24.7. $R$ being the identity reduction, (7) is precisely § 24.7, and from this special case the general case is securable by reduction in view of § 52*a*9, 4.

Further, (7) with §§ 54*a*$1_2$, 44*a*$2_1$ implies (8); (8, 5) implies (9); ($1_3$) with § 52*a*8 implies ($10_1$); ($10_1$) with § 44*a*$5_4$ implies ($10_2$); (9, $10_2$) implies (11), since $*$-extension leaves *contains* invariant; (6, 11) implies ($12_1$); ($12_1$) with § 44*a*$5_1$, $5_4$ implies ($12_2$).

55. *$*$-composite of a number of classes.*—Omitting now further consideration of reduction, in view of §§ 54*a*$1_3$; 44*a*$5_2$; 54*a*12, we have

THEOREM I. *$\mathfrak{M}'$, $\mathfrak{M}''$ being two classes of functions on $\mathfrak{P}'$, $\mathfrak{P}''$ respectively, having the dominance property $D$, the $*$-composite $(\mathfrak{M}'\mathfrak{M}'')_*$ is linear and closed, has the dominance property $D$, and is expressible in the forms:*

$$(\mathfrak{M}'\mathfrak{M}'')_* ; \quad (\mathfrak{M}'_*\mathfrak{M}'')_* ; \quad (\mathfrak{M}'\mathfrak{M}''_*)_* ; \quad (\mathfrak{M}'_*\mathfrak{M}''_*)_* .$$

The dominance property $D$ is invariant under $*$-extension and under multiplication of classes. Hence, as corollaries of the preceding theorem, we have the following theorems:

THEOREM II. *$\mathfrak{M}'$, $\mathfrak{M}''$, $\mathfrak{M}'''$ being three classes of functions on $\mathfrak{P}'$, $\mathfrak{P}''$, $\mathfrak{P}'''$ respectively, with the dominance property $D$, the $*$-composite $(\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''')_*$ is expressible in the various forms:*

$$(\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''')_*;\quad (\mathfrak{M}'_*\mathfrak{M}''\mathfrak{M}''')_*;\quad (\mathfrak{M}'\mathfrak{M}''_*\mathfrak{M}''')_*;\quad (\mathfrak{M}'\mathfrak{M}''\mathfrak{M}'''_*)_*,$$
$$(\mathfrak{M}'_*\mathfrak{M}''_*\mathfrak{M}''')_*;\quad (\mathfrak{M}'_*\mathfrak{M}''\mathfrak{M}'''_*)_*;\quad (\mathfrak{M}'\mathfrak{M}''_*\mathfrak{M}'''_*)_*;$$
$$(\mathfrak{M}'_*\mathfrak{M}''_*\mathfrak{M}'''_*)_*,$$

*derived from the original form by the $*$-extension of one or more of the constituent classes, and also in the various forms derived from these by the $*$-composition of pairs of constituent classes, e. g., from* $(\mathfrak{M}'_*\mathfrak{M}''_*\mathfrak{M}''')_*$ *the forms:*

$$((\mathfrak{M}'_*\mathfrak{M}''_*)_*\mathfrak{M}''')_*;\quad ((\mathfrak{M}'_*\mathfrak{M}''')_*\mathfrak{M}''_*)_*;\quad (\mathfrak{M}'_*(\mathfrak{M}''_*\mathfrak{M}''')_*)_*.$$

THEOREM III. $\mathfrak{M}', \cdots, \mathfrak{M}^{(n)}$ *being classes of functions on* $\mathfrak{P}', \cdots, \mathfrak{P}^{(n)}$ *respectively, with the dominance property $D$, the $*$-composite class:*

$$(\mathfrak{M}'\mathfrak{M}''\cdots\mathfrak{M}^{(n)})_*,$$

*is linear and closed and has the dominance property $D$. It is expressible in the various forms derivable from the original form by a finite sequence of operations of $*$-extension of individual constituents and of $*$-composition of two or more constituents of the original form or of a form already derived.*

THEOREM IV. *The genus $LCD$ of all classes $\mathfrak{M}$ linear, closed, and with dominance property $D$, is closed under $*$-composition of classes, that is,*

$$\mathfrak{M}'^{LCD}\,.\,\mathfrak{M}''^{LCD}\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')_*^{LCD}.$$

## *Characterization of the functions of the $*$-composite class $(\mathfrak{M}'\mathfrak{M}'')_*$.* § 56.

56. In the General Analysis of Functional Equations involving several independent variables the $*$-composite $(\mathfrak{M}'\mathfrak{M}'')_*$ of two classes $\mathfrak{M}'$, $\mathfrak{M}''$ enters fundamentally. For instance, the class $\mathfrak{C}$ of all continuous functions of two real variables: $0 \leqq p', p'' \leqq 1$, is the $*$-composite $(\mathfrak{C}'\mathfrak{C}'')_*$ of the classes $\mathfrak{C}'$, $\mathfrak{C}''$ of all continuous functions of the respective variables: $0 \leqq p' \leqq 1$, $0 \leqq p'' \leqq 1$.

There arises the necessity of conditioning the constituent classes $\mathfrak{M}'$, $\mathfrak{M}''$ of functions, perhaps with respect also to the constituent classes $\mathfrak{P}'$, $\mathfrak{P}''$ of elements, in such a way as to secure functional characterizations of the functions:

$$\rho \equiv (\rho_{p'p''} \mid p'p'') \equiv (\rho_p \mid p),$$

of the $*$-composite class:

$$\mathfrak{N} \equiv (\mathfrak{M}'\mathfrak{M}'')_*,$$

on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ to $\mathfrak{A}$.

In Part I we found that the classes $\mathfrak{M}^{\mathrm{I}}, \cdots, \mathfrak{M}^{\mathrm{IV}}$ have in common the closure and the dominance properties: $L$, $C$, $D$, $D_0$, $D_1$, $\Delta$. They belong to the genus $LCD$ of classes linear closed and with dominance property $D$; this genus is closed under $*$-composition of classes (§ 55 IV).

In Part II (§ 82 II) we are to secure conditions on the class $\mathfrak{M}'$ such that *for every class* $\mathfrak{M}''$ *of the genus* $LCD$ the $*$-composite $(\mathfrak{M}'\mathfrak{M}'')_*$ is the class $\mathfrak{F}$ of all functions $\phi$ on $\mathfrak{P}$ to $\mathfrak{A}$ satisfying certain two conditions:

$$(1^\circ)\ K'_{12}\,\mathfrak{M}'(\mathfrak{M}''); \qquad (2^\circ)\ \mathfrak{M}''(p').$$

(Cf. §§ 82.8, 72.12.) The second condition is that the function $\phi$ for every $p'$ belongs to the class $\mathfrak{M}''$. The first condition is defined in terms of $\mathfrak{M}'$, $\mathfrak{M}''$ and certain postulated features:

$$K'_1;\ K'_2,$$

of the class $\mathfrak{P}'$, viz., certain suitably conditioned relations: $K'_1$; $K'_2$, on the classes $\mathfrak{P}'\mathfrak{Z}$, $\mathfrak{P}'\mathfrak{P}'\mathfrak{Z}$ (where $\mathfrak{Z}$ is the class $[m]$ of positive integers), of which metrical instances are:

$$K'_1:\ K'_{p'm}\ .\equiv.\ Ap' \geqq m;$$

$$K'_2:\ K'_{p_1'p_2'm}\ .\equiv.\ A(p'_1 - p'_2) \leqq \frac{1}{m}.$$

In terms of the respective relations: $K'_1$; $K'_2$, we define (§ 72.7, 8) two properties:

$$K'_1;\ K'_2,$$

of classes $\mathfrak{M}'$ of functions on $\mathfrak{P}'$, and have the theorem that for every class $\mathfrak{M}'^{DK_1'K_2'}$ the class $\mathfrak{F}$ contains the class $(\mathfrak{M}'\mathfrak{M}'')_*$; cf. § 73.$2_{12}$, $5_{12}$.

On the other hand, we have the theorem (§ 81) that for every class $\mathfrak{M}'^{D\Delta'}$ the class $(\mathfrak{M}'\mathfrak{M}'')_*$ contains the class $\mathfrak{F}$, where the property $\Delta'$ of classes $\mathfrak{M}'$ on $\mathfrak{P}'$, and the relations: $K_1'$; $K_2'$, and so the property $K_{12}'\mathfrak{M}'(\mathfrak{M}'')$ of functions $\phi$ on $\mathfrak{P}$ are defined (§§ 75, 77, 79) in terms of a certain postulated development:

$$\Delta',$$

of the class $\mathfrak{P}'$ of elements, of which instances for the cases $\mathfrak{P}'^{\mathrm{III}}$, $\mathfrak{P}'^{\mathrm{IV}}$ are furnished by the denumeration of $\mathfrak{P}'^{\mathrm{III}}$ and the partition by sequential halving of the linear interval $\mathfrak{P}'^{\mathrm{IV}}$.

Accordingly, we have (§82.7, 8) the

THEOREM. *$\mathfrak{P}'$ being a class of elements and $\Delta'$ a development of $\mathfrak{P}'$,*

$$\mathfrak{M}'^{D\Delta' K_1'^{\Delta'} K_2'^{\Delta'}} . \mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{K'^{\Delta'}_{12}\mathfrak{M}'(\mathfrak{M}'') . \mathfrak{M}''(p')}].$$

Here the superscripts $\Delta'$ on the notations $K_1'$, $K_2'$, $K_{12}'$ are to denote that the corresponding properties are those defined in terms of the development $\Delta'$ of $\mathfrak{P}'$.

As leading up to the general theory which has been sketched, we shall consider (§§ 60–67) the cases III and IV in detail, after explaining (§§ 57–59) a scheme of notation useful in the study of functions of two independent variables.

## *Functions and classes of functions of two independent variables. General notations for properties involving uniformity and relativity.*

### §§ 57–59.

57. *Properties of functions of two variables. Uniformity.*— Denote functions to $\mathfrak{A}$ by the small Greek letters as usual, for instance,

$$\phi' \equiv (\phi'_{p'} \mid p'), \qquad \phi'' \equiv (\phi''_{p''} \mid p''), \qquad \phi \equiv (\phi_p \mid p) \equiv (\phi_{p'p''} \mid p'p'')$$

on the respective classes $\mathfrak{P}'$, $\mathfrak{P}''$, $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$.

$P'$ denoting a property of functions on $\mathfrak{P}'$, $\phi'^{P'}$ denotes a function $\phi'$ having the property $P'$.

A function $\phi \equiv (\phi_{p'p''} | p'p'')$ on $\mathfrak{P}'\mathfrak{P}''$ reduces for every $p''$ to a function $\phi_{p''} \equiv (\phi_{p'p''} | p')$ on $\mathfrak{P}'$ and for every $p'$ to a function $\phi_{p'} \equiv (\phi_{p'p''} | p'')$ on $\mathfrak{P}''$. (Cf. § 4.) In special cases, e.g., if $\mathfrak{P}' = \mathfrak{P}''$, the notations $\phi_{p'}$, $\phi_{p''}$ are obscure and require modification; in general, however, they will be found perspicuous. In case the context specifies the particular $p'$ or $p''$ we are at liberty to use instead of $\phi_{p'}$ or $\phi_{p''}$ the notation $\phi$ itself. Thus the statement; *for every $p''$ the function $\phi$ quâ function on $\mathfrak{P}'$ has the property $P'$*, we write in the three forms:

(1) $$p'' . \supset . \phi^{P'}; \quad \phi^{P'}(p''); \quad \phi^{P'(p'')},$$

of which the last two may be read: *the function $\phi$ has the property $P'$ for every $p''$*. Now the function $\phi$ may have the property $P'$ for every $p''$ of $\mathfrak{P}''$ and in some sense *uniformly** on $\mathfrak{P}''$; this we indicate in the two forms:

(2) $$\phi^{P'}(\mathfrak{P}''); \qquad \phi^{P'(\mathfrak{P}'')}.$$

For every particular use of this notation *the precise sense of the uniformity is to be defined*. Usually *uniformity enters with existential properties* and holds in case the objects (or some of the objects) specified as existing for every $p''$ of $\mathfrak{P}''$ exist independently of $p''$ on $\mathfrak{P}''$, i. e., uniformly on $\mathfrak{P}''$.

For example, setting for brevity,

$B_0 \equiv$ *is contained in*; $\quad B_1 \equiv$ *is dominated by*,

we give to the notations:

(3) $$\phi'^{B_0\mathfrak{M}'}; \qquad \phi^{B_0\mathfrak{M}'(p'')}; \qquad \phi^{B_0\mathfrak{M}'(\mathfrak{P}'')},$$

the respective meanings:

The function $\phi'$ belongs to the class $\mathfrak{M}'$;

For every $p''$ the function $\phi$, quâ function on $\mathfrak{P}'$, belongs to the class $\mathfrak{M}'$;

The function $\phi$, quâ for every $p''$ function on $\mathfrak{P}'$, is a function of the class $\mathfrak{M}'$ uniformly on $\mathfrak{P}''$, i. e., for the various $p''$ of $\mathfrak{P}''$ the same function of the class $\mathfrak{M}'$,

and we similarly give to the notations:

(4) $$\phi'^{B_1\mathfrak{M}'}; \qquad \phi^{B_1\mathfrak{M}'(p'')}; \qquad \phi^{B_1\mathfrak{M}'(\mathfrak{P}'')},$$

the respective meanings:

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* Cf. § 6.

The function $\phi'$ is dominated by the class $\mathfrak{M}''$, i. e., by some function $\mu''$ of $\mathfrak{M}''$.

For every $p''$ the function $\phi$, quâ function on $\mathfrak{P}'$, is dominated by the class $\mathfrak{M}''$, i. e., by some function $\mu''$ of $\mathfrak{M}''$ (perhaps varying with $p''$).

The function $\phi$, quâ for every $p''$ function on $\mathfrak{P}'$, is dominated by the class $\mathfrak{M}'$ uniformly on $\mathfrak{P}''$, i. e., by some function $\mu''$ of $\mathfrak{M}''$ the same for the various $p''$.

The definitions (3) generalize those of § 6.

57*a*. *Relativity.*—It is convenient to think of $P'(p'')$ and $P'(\mathfrak{P}'')$ as properties of functions on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$. Similarly, with reference to a property $P'$ and a class $\mathfrak{M}''$ it is frequently convenient to define a property $P'(\mathfrak{M}'')$ of functions on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$. This property $P'(\mathfrak{M}'')$ is *relative to* the class $\mathfrak{M}''$ as *scale;* in practice, it is *relative uniformity.*

For example, let $\mathfrak{P}'$ be $\mathfrak{P}'^{\text{IV}}: a'_0 \leqq p' \leqq a'_1$, and let $P'$ denote *uniformly continuous on* $\mathfrak{P}'$, viz., let $\phi'^{P'}$ mean

$$(1) \quad e : \supset : \exists\, d_e \ni A(p'_1 - p'_2) < d_e \,.\supset .\, A(\phi'_{p_1'} - \phi'_{p_2'}) \leqq e.$$

Then $\phi^{P'(p'')}$ means

$$(2) \quad \begin{aligned} & p'' \therefore \supset \therefore e : \supset : \exists\, d_{p''e} \ni \\ & \qquad A(p'_1 - p'_2) < d_{p''e} \,.\supset .\, A(\phi_{p_1'p''} - \phi_{p_2'p''}) \leqq e, \end{aligned}$$

and we give to $\phi^{P'(\mathfrak{P}'')}$, $\phi^{P'(\mathfrak{M}'')}$ the respective meanings:

$$(3) \quad \begin{aligned} & e \therefore \supset \therefore \exists\, d_e \ni \\ & \qquad A(p'_1 - p'_2) < d_e : \supset : p'' \,.\supset .\, A(\phi_{p_1'p''} - \phi_{p_2'p''}) \leqq e\,; \end{aligned}$$

$$(4) \quad \begin{aligned} & \exists\, \mu'' \ni e \therefore \supset \therefore \exists\, d_e \ni \\ & \qquad A(p'_1 - p'_2) < d_e : \supset : p'' \,.\supset .\, A(\phi_{p_1'p''} - \phi_{p_2'p''}) \leqq eA\mu''_{p''}. \end{aligned}$$

In accordance with § 6, omitting $p''$ and understanding it as implied by the notations involving $\phi$, we may write (3; 4) in the compacter forms:*

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*3′) For every positive number $e$ there exists a positive number $d_e$ (dependent on $e$) such that for every two (real numbers) elements $p_1'$, $p_2'$ of $\mathfrak{P}'^{\text{IV}}$ the absolute value of whose difference is less than $d_e$ the absolute of the difference of the corresponding functions $\phi_{p_1'}$, $\phi_{p_2'}$ is (for every $p''$) at most $e$.

$$(3') \quad e : \supset : \exists\ d_e \ni A(p_1' - p_2') < d_e \,.\supset .\, A(\phi_{p_1'} - \phi_{p_2'}) \leqq e\,;$$

$$(4') \quad \exists\ \mu'' \ni e : \supset : \exists\ d_e \ni \quad A(p_1' - p_2') < d_e \,.\supset .\, A(\phi_{p_1'} - \phi_{p_2'}) \leqq e A\mu''.$$

In the case (3) $\phi$ is said to be *uniformly continuous on* $\mathfrak{P}'$, *uniformly on* $\mathfrak{P}''$, and in the case (4) $\phi$ is said to be *uniformly continuous* on $\mathfrak{P}'$, *uniformly as to scale* $\mathfrak{M}''$ *on* $\mathfrak{P}''$. We shall prove below (§§ 62, 66) that, if $\mathfrak{C}'$ is the class of functions continuous and so uniformly continuous on $\mathfrak{P}'$, and $\mathfrak{M}''$ is a class, linear closed and with dominance property $D$, of functions on $\mathfrak{P}''$, the class $(\mathfrak{C}'\mathfrak{M}'')_*$ is the class of all functions $\phi$ of $p'p''$ such that $\phi$ for every $p'$ belongs to $\mathfrak{M}''$ and for every $p''$ belongs to $\mathfrak{C}'$, its uniform continuity on $\mathfrak{P}'$ being uniform as to $\mathfrak{M}''$ on $\mathfrak{P}''$, that is, in symbols,

$$(5) \qquad (\mathfrak{C}'\mathfrak{M}'')_* = [\text{all } \phi^{P'(\mathfrak{M}'') \cdot B_0\mathfrak{M}''(p')}],$$

$P'$ denoting *uniformly continuous on* $\mathfrak{P}'$.

For a property $P'$, the notions $P'(\mathfrak{P}'')$ and $P'(\mathfrak{M}'')$ are usually determined, as in the preceding instance, in such a way as to satisfy the conditions:

$$A) \qquad \mathfrak{M}''^{B_{\mathfrak{P}''}} : \supset : \phi^{P'(\mathfrak{M}'')} \,.\supset .\, \phi^{P'(\mathfrak{P}'')}\,;$$

$$B) \qquad \phi^{P'(\mathfrak{M}'')} \,.\supset .\, \phi^{P'(p'')}\,;$$

$$C) \qquad \phi^{P'(\mathfrak{A}\mathfrak{P}'')} \,.\supset .\, \phi^{P'(\mathfrak{P}'')}.$$

58. *Properties of classes of functions of two variables.*— The notational conventions of § 57 are extensible to classes $\mathfrak{K}'$, $\mathfrak{K}''$, $\mathfrak{K}$ of functions on $\mathfrak{P}'$, $\mathfrak{P}''$, $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ and properties $P'$ of classes $\mathfrak{K}'$ on $\mathfrak{P}'$. Thus enter the notations:

$$\mathfrak{K}'^{P'}\,;\ \mathfrak{K}^{P'(p'')}\,;\ \mathfrak{K}^{P'(\mathfrak{P}'')}\,;\ \mathfrak{K}^{P'(\mathfrak{M}'')}.$$

59. *Derivation of properties.*— From a property $P$ of functions on $\mathfrak{P}$ we derive (cf. § 2) a property $P$ of classes of functions: the class $\mathfrak{M}$ has the property $P$ (of classes) if its every function $\mu$ has the property $P$ (of functions), i. e.,

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4′) There exists a function $\mu''$ of the class $\mathfrak{M}''$ such that for every positive number $e$ · · the absolute of the difference of the corresponding functions $\phi_{p_1'}$, $\phi_{p_2'}$ is at most $e$ multiplied by the absolute of $\mu''$ (for every $p''$).

$$(1) \qquad \mathfrak{M}^{P} :\equiv: \mu \,.\supset.\, \mu^{P}.$$

Thus, from the properties:

(2) $P$: *is contained in* $\mathfrak{S}$; *is dominated by* $\mathfrak{S}$,

of functions we derive properties similarly expressed of classes, in accordance with the terminology used in Part I.

Under this derivation of properties equivalence and implication of properties are invariant, while negation is not invariant.

The use in this way of the same notation $P$ for closely related properties of functions and of classes, according to the context, facilitates the argumentation and in practice causes no confusion.

In general, *in the interpretation of notations one is to have regard to the chronologies.* Thus, whether $P'$ is a property of functions or of classes on $\mathfrak{P}'$, we have agreed to denote by $P'(p'')$ a definite property of functions or of classes on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$. Hence the notation:

$$(3) \qquad \mathfrak{F}^{P'(p'')},$$

$P'$ being primarily a property of functions on $\mathfrak{P}'$, is interpretable in the two senses:

$$(4) \qquad \phi^{\mathfrak{F}} \,.\supset.\, \phi^{P'(p'')};$$

$$(5) \qquad p'' \,.\supset.\, \mathfrak{F}^{P'},$$

according as the notation $P'(p'')$ is understood to denote primarily a property of functions or of classes. The two interpretations have however the same meaning:

$$(6) \qquad \phi^{\mathfrak{F}} . p'' \,.\supset.\, \phi^{P'}.$$

But no general definition of the notations; $P'(\mathfrak{P}'')$, $P'(\mathfrak{M}'')$, as properties of functions or of classes on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ has been given. We agree that in defining the notations primarily as properties of functions we define them at once secondarily as the derived property of classes, so that the notations:

$$(7) \qquad \mathfrak{F}^{P'(\mathfrak{P}'')}; \qquad \mathfrak{F}^{P'(\mathfrak{M}'')},$$

have the respective interpretations:

$$(8) \qquad \phi^{\mathfrak{F}} \,.\supset.\, \phi^{P'(\mathfrak{P}'')}; \qquad \phi^{\mathfrak{F}} \,.\supset.\, \phi^{P'(\mathfrak{M}'')}.$$

*rties $B$ bipartite on functions and classes of functions. The*
*red properties $B$, $B_*$ of classes of functions.* §§ 60–63.

*Bipartite properties $B$.* — In the functional characteriza-
to be secured there enter fundamentally properties $B$
ite on (or relations $B$ between) functions and classes of
ons on $\mathfrak{P}$. We use the various notations:

$$(\phi, \mathfrak{M})^B \quad \text{or} \quad \phi B \mathfrak{M} \quad \text{or} \quad \phi^{B\mathfrak{M}}.$$

h a bipartite property $B$ gives rise in connection with
class $\mathfrak{M}$ to a property $B\mathfrak{M}$ of functions $\phi$, and the third
on (1) expresses this fact. (We do not now need to at-
o the property $\phi B$ of classes of functions.) Thus, in
lance with preceding conventions, we meet the notations:

$$\phi'^{B'\mathfrak{M}'}, \quad \phi^{B'\mathfrak{M}'(p'')}, \quad \phi^{B'\mathfrak{M}'(\mathfrak{P}'')}, \quad \phi^{B'\mathfrak{M}'(\mathfrak{M}'')};$$
$$\widetilde{\mathfrak{M}}'^{B'\mathfrak{M}'}, \quad \widetilde{\mathfrak{M}}^{B'\mathfrak{M}'(p'')}, \quad \widetilde{\mathfrak{M}}^{B'\mathfrak{M}'(\mathfrak{P}'')}, \quad \widetilde{\mathfrak{M}}^{B'\mathfrak{M}'(\mathfrak{M}'')},$$

$B'$ is a property bipartite on functions and classes of
ons on $\mathfrak{P}'$, while $B'\mathfrak{M}'(\mathfrak{P}'')$, $B'\mathfrak{M}'(\mathfrak{M}'')$ are properties of
ons on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$, subject to definition for the various
rties $B'$ to be considered.

example, the bipartite properties $B$:

$B_0$: *is contained in;* $\quad$ $B_1$: *is dominated by,*

the properties $B\mathfrak{M}$:

$B_0\mathfrak{M}$: *is contained in* $\mathfrak{M}$; $\quad$ $B_1\mathfrak{M}$: *is dominated by* $\mathfrak{M}$,

rily of functions and secondarily of classes of functions on
the respective senses already used. The corresponding
rties* $B_0\mathfrak{M}'$, $B_1\mathfrak{M}'$ are the properties of § 57.3$_1$, 4$_1$
tively. Thus the properties:

$$B_0\mathfrak{M}'(\mathfrak{P}''); \qquad B_1\mathfrak{M}'(\mathfrak{P}''),$$

ready defined (§ 57.3, 4). We define

$$B_0\mathfrak{M}'(\mathfrak{M}''); \qquad B_1\mathfrak{M}'(\mathfrak{M}'')$$

noting

$$B_0\mathfrak{M}'\mathfrak{M}''; \qquad B_1\mathfrak{M}'\mathfrak{M}'',$$

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[*] ne properties $B_0$, $B_1$ are of general reference, and for that reason we do
ite $B_0'$, $B_1'$.

and, as second definitions, for temporary use in the examples of § 63, we define

$$(7^{(2)}) \qquad B_0^{(2)}\mathfrak{M}'(\mathfrak{M}''); \qquad B_1^{(2)}\mathfrak{M}'(\mathfrak{M}'')$$

as denoting

$$(8^{(2)}) \qquad B_0\mathfrak{M}'(\mathfrak{P}''); \qquad B_1\mathfrak{M}'(\mathfrak{P}'').$$

61. *The derived properties $B$, $B_*$ of classes of functions.*—A property $B'$ binary on functions and classes of functions on $\mathfrak{P}'$ gives rise to two properties:

$$(1) \qquad B', \ B'_*$$

of classes $\mathfrak{M}'$ of functions on $\mathfrak{P}'$.

*A class $\mathfrak{M}'$ has the property $B'$* in case its every function $\mu'$ has with $\mathfrak{M}'$ the bipartite property $B'$, that is,

$$(2) \qquad \mathfrak{M}'^{B'} : \equiv : \mathfrak{M}' \ni \mu' . \supset . \mu'^{B'\mathfrak{M}'},$$

so that

$$(3) \qquad \mathfrak{M}'^{B'} . \sim . \mathfrak{M}'^{B'\mathfrak{M}'}.$$

For example, we have the proposition:

$$\mathfrak{M}' . \supset . \mathfrak{M}'^{B_0B_1},$$

that is, every class $\mathfrak{M}'$ is contained in and is dominated by itself.

*A class $\mathfrak{M}'$ has the property $B'_*$* in case for every class $\mathfrak{M}''^{LCD}$ on every class $\mathfrak{P}''$ the $*$-composite class $(\mathfrak{M}'\mathfrak{M}'')_*$ on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ may be characterized as the class of all functions $\phi$ on $\mathfrak{P}$ having the properties:

$$(4) \qquad B'\mathfrak{M}'(\mathfrak{M}''); \qquad B_0\mathfrak{M}''(p'),$$

that is,

$$(5) \qquad \mathfrak{M}'^{B'_*} : \equiv : \mathfrak{M}' \ni \mathfrak{M}''^{LCD} . \supset . (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{B'\mathfrak{M}'(\mathfrak{M}'') . B_0\mathfrak{M}''(p')}].$$

Since $\mathfrak{M}''$ is linear and closed, so that $\mathfrak{M}''_* = \mathfrak{M}''$, it is evident that every function of $(\mathfrak{M}'\mathfrak{M}'')_*$ has for every $p'$ the property $B_0\mathfrak{M}''$: *is contained in* $\mathfrak{M}''$.

For a particular bipartite property $B'$ there is for every determination of the property $B'\mathfrak{M}'(\mathfrak{M}'')$ of functions on $\mathfrak{P}$ a precise determination of the property $B'_*$ of classes of functions on $\mathfrak{P}'$, and accordingly of the definition of the classes $\mathfrak{M}'$ having the property $B'_*$. By way of illustration I consider in § 63 the properties $B_{0*}$, $B_{1*}$, $B_0^{(2)}{}_*$, $B_1^{(2)}{}_*$.

It is evident that a property $B'_*$ may be possessed by no (existing) class $\mathfrak{M}'$. For instance, $B'$ denoting *is not contained in*, and $B'\mathfrak{M}'(\mathfrak{M}'')$ denoting *is not contained in* $\mathfrak{M}'\mathfrak{M}''$, there exists no class $\mathfrak{M}'$ with the property $B'_*$, since every class $(\mathfrak{M}'\mathfrak{M}'')_*$ contains $\mathfrak{M}'\mathfrak{M}''$.

However, as we shall see, the classes $\mathfrak{M}$ possessing some property $B_*$ are very numerous, including, in particular, the classes:

$$\mathfrak{M}^{\mathrm{I}};\ \mathfrak{M}^{\mathrm{II}_a};\ \mathfrak{M}^{\mathrm{III}};\ \mathfrak{M}^{\mathrm{III}_b};\ \mathfrak{M}^{\mathrm{III}_c};\ \mathfrak{M}^{\mathrm{IV}},$$

and classes arising from them by sequential $*$-composition. Cf. §§ 651, 661, 84.

61*a*. *Propositions.*—As corollaries of the definitions of properties $B$, $B_*$, etc., we notice the propositions:*

* 1) If $\mathfrak{M}'$ on $\mathfrak{P}'$ and $\mathfrak{M}''$ on $\mathfrak{P}''$ are two classes linear, closed, having the dominance property $D$ and with the respective properties $B'_*$ and $B''_*$, then every function $\varphi$ on $\mathfrak{P}\ \ \mathfrak{P}'\mathfrak{P}''$ with the properties: $B'\mathfrak{M}'(\mathfrak{M}'')$; $B_0\mathfrak{M}''(p')$, belongs to $(\mathfrak{M}'\mathfrak{M}'')_*$ and has the properties: $B''\mathfrak{M}''(\mathfrak{M}')$: $B_0\mathfrak{M}'(p'')$, and conversely, every function $\varphi$ having the properties $B''\mathfrak{M}''(\mathfrak{M}')$; $B_0\mathfrak{M}'(p'')$, belongs to $(\mathfrak{M}'\mathfrak{M}'')_*$ and has the properties: $B'\mathfrak{M}'(\mathfrak{M}'')$; $B_0\mathfrak{M}''(p')$.

2′) In case the properties $B'$, etc., are such that

(*a*) for every function $\varphi'$ and class $\mathfrak{M}'$ on $\mathfrak{P}'$ the properties $B'\mathfrak{M}'(\mathfrak{A})$ and $B'\mathfrak{M}'$ of $\varphi'$ are equivalent; and

(*b*) for every function $\varphi$ on $\mathfrak{P}\ \ \mathfrak{P}'\mathfrak{P}''$ and pair of classes $\mathfrak{M}'$ and $\mathfrak{S}'$ on $\mathfrak{P}'$, $\mathfrak{M}'$ being dominated by $\mathfrak{A}\mathfrak{S}'$, and class $\mathfrak{M}''^{LCD}$ on $\mathfrak{P}''$ the property $B'\mathfrak{M}'(\mathfrak{M}'')$ of $\varphi$ implies the property $B'\mathfrak{S}'(\mathfrak{M}'')$ of $\varphi$,

it follows that

(*c*) if $\mathfrak{M}'$ has the property $B'_*$, then $\mathfrak{M}'_*$ is the class of all functions $\varphi'$ with the property $B'\mathfrak{M}'$, and $\mathfrak{M}'_*$ has the property $B'\mathfrak{M}'$, and $\mathfrak{M}'$ has the property $B'$, and accordingly the two properties: $B_0\mathfrak{M}'_*$; $B'\mathfrak{M}'$, of functions $\varphi'$ are equivalent; and

(*d*) if $\mathfrak{M}'$ has the dominance property $D$ and the property $B'_*$, then $\mathfrak{M}'$ has the property $B'$, and $\mathfrak{M}'_L$ is linear and has the dominance property $D$ and the properties: $B'$; $B'_*$; $B_1\mathfrak{A}\mathfrak{M}'$, and $\mathfrak{M}'_*$ is linear and closed and has the dominance property $D$ and the properties: $B'$; $B'_*$; $B_1\mathfrak{A}\mathfrak{M}'$; and accordingly,

(*e*) if $\mathfrak{M}'$ has the dominance property $D$, then the property $B'_*$ belongs to both or to neither of the classes: $\mathfrak{M}'$; $\mathfrak{M}'_*$, and

(*f*) if $\mathfrak{M}'$ is linear and closed and has the property $B'_*$, then $\mathfrak{M}'$ has the property $B'$, and the properties: $B_0\mathfrak{M}'$; $B'\mathfrak{M}'$, of functions $\varphi'$ are equivalent.

$$\mathfrak{M}'^{LCDB'*} . \mathfrak{M}''^{LCDB''*} : \supset :$$

1) $$(\phi^{B'\mathfrak{M}'(\mathfrak{M}'') . B_0\mathfrak{M}''(p')} . \supset . \phi^{B''\mathfrak{M}''(\mathfrak{M}') . B_0\mathfrak{M}'(p'') . B_0(\mathfrak{M}'\mathfrak{M}'')_*}).$$

$$(\phi^{B''\mathfrak{M}''(\mathfrak{M}') . B_0\mathfrak{M}'(p'')} . \supset . \phi^{B'\mathfrak{M}'(\mathfrak{M}'') . B_0\mathfrak{M}''(p') . B_0(\mathfrak{M}'\mathfrak{M}'')_*}).$$

2) In case the properties $B'$, etc., are such that

(2a) $$\phi' . \mathfrak{M}' : \supset : \phi'^{B'\mathfrak{M}'(\mathfrak{A})}_{.} . \sim . \phi'^{B'\mathfrak{M}'} ;$$

(2b) $$\phi . \mathfrak{M}'^{B_1\mathfrak{A}\mathfrak{S}'} . \mathfrak{M}''^{LCD} : \supset : \phi^{B'\mathfrak{M}'(\mathfrak{M}'')} . \supset . \phi^{B'\mathfrak{S}'(\mathfrak{M}'')},$$

it follows that

(2c) $$\mathfrak{M}'^{B'*} . \supset . \mathfrak{M}'_* = [\text{all}\, \phi'^{B'\mathfrak{M}'}] . \mathfrak{M}'^{B'\mathfrak{M}'}_* . \mathfrak{M}'^{B_1} . (B_0\mathfrak{M}'_* \sim B'\mathfrak{M}');$$

(2d) $$\mathfrak{M}'^{DB'*} . \supset . \mathfrak{M}'^{B'} . \mathfrak{M}'^{LDB'B'* . B_1\mathfrak{A}\mathfrak{M}'}_L . \mathfrak{M}'^{LCDB'B'* . B_1\mathfrak{A}\mathfrak{M}'}_* ;$$

(2e) $$\mathfrak{M}'^{D} : \supset : \mathfrak{M}'^{B'*} . \sim . \mathfrak{M}'^{B'*}_* ;$$

(2f) $$\mathfrak{M}'^{LCB'*} . \supset . \mathfrak{M}'^{B'} . (B_0\mathfrak{M}' \sim B'\mathfrak{M}').$$

The first proposition is the basis of a body of theorems concerning functions of two variables falling in many cases under the general head of the interchange of order of limiting processes. $B''$ denotes a property bipartite on functions and classes of functions on $\mathfrak{P}''$. $B'$ and $B''$ may be instances in the cases $\mathfrak{P}'$ and $\mathfrak{P}''$ of properties of general reference, that is, like $B_1$: *is dominated by,* applicable in general; that they are such instances is not assumed; one or both may be properties of special reference,— to the class $\mathfrak{P}'$ or $\mathfrak{P}''$ of elements or to any other features of a particular case.

As to the second proposition, we notice that the conditions ($2a$, $b$) on $B'$, etc., implying (cf. $2f$) for classes $\mathfrak{M}'^{LCB'*}$ the equivalence of the properties : $B_0\,\mathfrak{M}'$; $B'\,\mathfrak{M}'$, of functions $\phi'$ on $\mathfrak{P}'$ do not imply for such classes $\mathfrak{M}'$ the equivalence of the properties: $B_0\,\mathfrak{M}'(\mathfrak{M}'')$; $B'\,\mathfrak{M}'(\mathfrak{M}'')$, and so do not imply that such classes $\mathfrak{M}'$ have the property $B_{0*}$.

62. *An example.*—In order to recognize that *the functional characterization of* $(\mathfrak{C}'\mathfrak{M}'')_*$ *given in* § 57$a$5 *is an instance of the functional characterization of* $(\mathfrak{M}'\mathfrak{M}'')_*$ *given in* § 61.5, we need merely to notice that the property $P'$ of functions on $\mathfrak{P}'$ (§ 57$a$1)

is for the case $\mathfrak{M}' \equiv \mathfrak{C}'$ the property $K_2'\mathfrak{M}'$ arising from a property* $K_2'$ bipartite on functions and classes of functions on $\mathfrak{P}' \equiv \mathfrak{P}'^{\mathrm{IV}}$, viz., from the property $K_2'$ for which $\phi' K_2' \mathfrak{M}'$ or $\phi'^{K_2'\mathfrak{M}'}$ means

$$(1) \qquad \exists\ \mu' \ni e : \supset : \exists\ d_e \ni \qquad A(p_1' - p_2') \leqq d_e . \supset . A(\phi'_{p'_1} - \phi'_{p'_2}) \leqq eA\mu'_{p'_1} .$$

For this property $K_2'$ we give to $\phi^{K_2'\mathfrak{M}'(\mathfrak{M}'')}$ the meaning:

$$(2) \qquad \exists\ \mu'' \ni \exists\ \mu' \ni e : \supset : \exists\ d_e \ni \qquad A(p_1' - p_2') \leqq d_e . \supset . A(\phi_{p'_1} - \phi_{p'_2}) \leqq eA\mu'_{p'_1}\mu'' .$$

These properties: $B'\mathfrak{M}'$, $B'\mathfrak{M}'(\mathfrak{M}'')$, for $B' = K_2'$, satisfy the respective relations:

$$A) \qquad \begin{gathered} \mathfrak{M}'^{B_1\mathfrak{C}'} : \supset : \phi'^{B'\mathfrak{M}'} . \supset . \phi'^{B'\mathfrak{C}'} ; \\ \mathfrak{M}'^{B_1\mathfrak{C}'} . \mathfrak{M}''^{B_1\mathfrak{C}''} : \supset : \phi^{B'\mathfrak{M}'(\mathfrak{M}'')} . \supset . \phi^{B'\mathfrak{C}'(\mathfrak{C}'')} . \end{gathered}$$

Now the properties $P'$, $P'(\mathfrak{M}'')$ of § 57*a*1, 4 are for the case $\mathfrak{M}' \equiv 1(\mathfrak{P}')$ (in which $\mu' = 1$ is the only function of $\mathfrak{M}'$) the properties $K'\mathfrak{M}'$, $K'\mathfrak{M}'(\mathfrak{M}'')$ just defined; and, by the remark $A$), for the two cases: $\mathfrak{M}' \equiv 1(\mathfrak{P}')$; $\mathfrak{C}'$, the two properties $K'\mathfrak{M}'$ are equivalent and the two properties $K'\mathfrak{M}'(\mathfrak{M}'')$ are equivalent. Thus we may rewrite § 57*a*5 in the form:

$$(3) \qquad (\mathfrak{C}'\mathfrak{M}'')_* = \lfloor \text{all } \phi^{K_2'\mathfrak{C}'(\mathfrak{M}'') . B_0\mathfrak{M}''(\mathfrak{P}')}] .$$

Accordingly, as we shall prove in § 66, *the class* $\mathfrak{M}^{\mathrm{IV}}$ *has the property* $K_{2*}$.

63. *Other examples.*—For the bipartite properties $B$:

$B_0$: *is contained in*; $\qquad B_1$: *is dominated by*,

of § 60 the notations $B\mathfrak{M}'(\mathfrak{M}'')$, $B^{(2)}\mathfrak{M}'(\mathfrak{M}'')$ have been defined (§ 60.8, $8^{(2)}$). We consider the corresponding properties $B_* = B_{0*}, B_{1*}, B_0^{(2)}{}_*, B_1^{(2)}{}_*$.

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* This bipartite property $K_2'$, quâ property on functions and classes of functions on the special class $\mathfrak{P}' \equiv \mathfrak{P}'^{\mathrm{IV}}$, is of special reference. It is later developed (cf. §§ 66*b*), 67, 72, 75, 77) into a property $K_2'$ of more general reference, viz. to any class $\mathfrak{P}'$ with development $\Delta'$, the notion *development* $\Delta'$ of $\mathfrak{P}'$ (§ 75) being a generalization of the partition by sequential halving of the linear interval $\mathfrak{P}'^{\mathrm{IV}}$.

1) $$\mathfrak{M}'^{B_0^{(2)}{}_*} \,.\sim.\, \mathfrak{M}' = 0(\mathfrak{P}'),$$

viz., the class $\mathfrak{M}'$ on $\mathfrak{P}'$ has the property $B_0^{(2)}{}_*$ if and only if it consists of the single function $0(\mathfrak{P}')$.

For $\mathfrak{M}'^{B_0^{(2)}{}_*}$ with $\mathfrak{M}''^{LCD}$ implies

$$(\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{B_0^{(2)}\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{M}''(p')}] = [\text{all } \phi^{B_0\mathfrak{M}'(\mathfrak{P}'') \,.\, B_0\mathfrak{M}''(p')}].$$

Now $\phi^{B_0\mathfrak{M}'(\mathfrak{P}'')}$ implies $\phi^{B_0\mathfrak{A}(\mathfrak{P}'')(p')}$.

Take $\mathfrak{P}'' \equiv \mathfrak{P}''^{\mathrm{II}_2}$ and $\mathfrak{M}'' \equiv \mathfrak{M}''^{\mathrm{II}_2}$. Then $\mathfrak{M}''^{LCD}$. Moreover $\mathfrak{M}''$ contains functions not of $\mathfrak{A}(\mathfrak{P}'')$. Let $\mu_0''$ be such a function; then $a\mu_0''$ is of $\mathfrak{A}(\mathfrak{P}'')$ only if $a = 0$. Now for every $\mu'$ of $\mathfrak{M}'$ $\mu'\mu_0''$ is of $\mathfrak{M}'\mathfrak{M}''$ and so of $(\mathfrak{M}'\mathfrak{M}'')_*$ and so of $\mathfrak{A}(\mathfrak{P}'')$ $(p')$; hence $\mu' = 0$ $(p')$. Accordingly, $\mathfrak{M}'^{B_0^{(2)}{}_*}$ implies $\mathfrak{M}' = 0(\mathfrak{P}')$. The converse implication is evident.

2) $$\mathfrak{M}'^{B_1^{(2)}{}_*} \,.\sim.\, \mathfrak{M}' = 0(\mathfrak{P}').$$

For $\mathfrak{M}'^{B_1^{(2)}{}_*}$ with $\mathfrak{M}''^{LCD}$ implies

$$(\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{B_1^{(2)}\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{M}''(p)}] = [\text{all } \phi^{B_1\mathfrak{M}'(\mathfrak{P}'') \,.\, B_0\mathfrak{M}''(p')}],$$

and accordingly $(\mathfrak{M}'\mathfrak{M}'')^{B_1\mathfrak{M}'(\mathfrak{P}'')}$, and so, unless $\mathfrak{M}' = 0(\mathfrak{P}')$, $\mathfrak{M}''^{B_1\mathfrak{A}(\mathfrak{P}'')}$. Now $\mathfrak{M}'' \equiv \mathfrak{M}''^{\mathrm{III}} \equiv$ the class of all functions on $\mathfrak{P}''^{\mathrm{III}}$ has the properties $LCD$ but not the property $B_1\mathfrak{A}(\mathfrak{P}'')$. Hence $\mathfrak{M}'^{B_1^{(2)}{}_*}$ implies $\mathfrak{M}' = 0(\mathfrak{P})$. The converse implication is evident.

3) $$\mathfrak{M}'^{B_{0*}} \,.\sim.\, \mathfrak{M}' \ni (\mathcal{E}\ (\mathfrak{A}_0^{\mathfrak{A}} \,.\, \mu_0'^{\mathfrak{M}'}) \ni \mathfrak{M}' = \mathfrak{A}_0\mu_0'),$$

viz., the class $\mathfrak{M}'$ on $\mathfrak{P}'$ has the property $B_{0*}$ if and only if it has the form: $\mathfrak{A}_0\,\mu_0'$, where $\mathfrak{A}_0$ is a class of real numbers and $\mu_0'$ is a function of $\mathfrak{M}'$.

In this case $\mathfrak{M}'^{B_{0*}}$ with $\mathfrak{M}''^{LCD}$ implies

$$(\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{B_0\mathfrak{M}'\mathfrak{M}'' \,.\, B_0\mathfrak{M}''(p')}] = [\text{all } \phi^{B_0\mathfrak{M}'\mathfrak{M}''}],$$

that is, $(\mathfrak{M}'\mathfrak{M}'')_* = \mathfrak{M}'\mathfrak{M}''$, so that $\mathfrak{M}'_* = \mathfrak{M}'$. We prove that in $\mathfrak{M}'$ there is a function $\mu_0'$ of which every function $\mu'$ is a numerical multiple. Consider any two functions $\mu_1'$, $\mu_2'$ of $\mathfrak{M}'$. Then, taking $\mathfrak{M}'' = \mathfrak{M}''^{\mathrm{II}_2}$, since $(\mathfrak{M}'\mathfrak{M}'')_L = \mathfrak{M}'\mathfrak{M}''$, we see that $\mu_1'(1, 0) + \mu_2'(0, 1) = \mu_3'(a_1, a_2)$, that is, there is a function $\mu_3'$ of which $\mu_1'$ and $\mu_2'$ are numerical multiples: $\mu_1' = a_1\mu_3'$, $\mu_2' = a_2\mu_3'$.

Hence, either $a_1 = 0$, so that $\mu'_1 = 0(\mathfrak{P}')$, or every function $\mu'_2$ is a numerical multiple of $\mu'_1$: $\mu'_2 = \mu'_1 a_2/a_1$. Thus, $\mathfrak{M}' = \mathfrak{A}_0 \mu'_0$ where $\mu'_0$ is any function of $\mathfrak{M}'$ not everywhere 0, or, if there is no such function, $\mu'_0 = 0(\mathfrak{P}')$ is the only function; hence, in either event $\mathfrak{M}'$ has the form $\mathfrak{A}_0 \mu'_0$. Conversely, every class $\mathfrak{M}'$ of the form $\mathfrak{A}_0 \mu'_0$ has the property $B_{0*}$.

4) $\mathfrak{M}'^{B_{1*}}$ with $\mathfrak{M}''^{LCD}$ implies

$$(\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{B_1 \mathfrak{M}'\mathfrak{M}'' . B_0 \mathfrak{M}''(p')}],$$

and accordingly

4a) $$\mathfrak{M}'_* = [\text{all } \phi'^{B_1 \mathfrak{A}\mathfrak{M}'}]; \ \mathfrak{M}'^{B_1 \mathfrak{A}\mathfrak{M}'}_*.$$

Hence, since $\mathfrak{M}'_*$ contains $\mathfrak{A}\mathfrak{M}'$,

4b) $$\phi^{B_1 \mathfrak{M}'\mathfrak{M}'' . B_0 \mathfrak{M}''(p')} . \sim . \phi^{B_1 \mathfrak{M}'_* \mathfrak{M}'' . B_0 \mathfrak{M}''(p')};$$

4c) $\mathfrak{M}'^{B_{1*}} . \mathfrak{M}'^{B_{1*}}_* . \supset . (\mathfrak{M}'\mathfrak{M}'')_* = (\mathfrak{M}'_* \mathfrak{M}'')_* . \mathfrak{M}'_* = \mathfrak{M}'_{**}$.

Further, by § 54*a*12 and § 44*a*I, we see that

4d) $$\mathfrak{M}'^{D} : \supset : \mathfrak{M}'^{B_{1*}} . \sim . \mathfrak{M}'^{B_{1*}}_*;$$

4e) $\mathfrak{M}'^{LCDB_{1*}} : \supset : \mathfrak{M}'_* = \mathfrak{M}' = [\text{all } \phi'^{B_1 \mathfrak{M}}] : \phi'^{B_1 \mathfrak{M}'} . \sim . \phi'^{B_0 \mathfrak{M}'}$,

that is, in effect, (*d*) a class $\mathfrak{M}'$ having the dominance property $D$, the classes $\mathfrak{M}'$, $\mathfrak{M}'_*$ either both have or both do not have the property $B_{1*}$; (*e*) a class $\mathfrak{M}'$ linear closed and having the dominance property $D$ and the property $B_{1*}$ contains every function it dominates.

We shall prove in § 65 that *the classes*

$$\mathfrak{M}' = \mathfrak{M}'^{\mathrm{I};\ \mathrm{II}_n;\ \mathrm{III};\ \mathrm{III}_0;\ \mathrm{III}_c}$$

*have the property* $B_{1*}$, while by 4*e* the class $\mathfrak{M}'^{\mathrm{IV}}$ does not have the property $B_{1*}$.

### *Classes* $\mathfrak{F}$ *containing* $(\mathfrak{M}'\mathfrak{M}'')_*$. § 64.

64. THEOREM. *The respective hypotheses:*

(1) $$\mathfrak{M}'^{D_1} . \mathfrak{M}''^{D_1} . \mathfrak{F} \equiv [\text{all } \phi^{B_1 \mathfrak{A}\mathfrak{M}'\mathfrak{M}''}];$$

(2) $$\mathfrak{M}' . \mathfrak{M}''^{LCD_1} . \mathfrak{F} \equiv [\text{all } \phi^{B_0 \mathfrak{M}''(p')}];$$

(3) $$\mathfrak{M}'^{LCD_1} . \mathfrak{M}''^{LCD_1} . \mathfrak{F} \equiv [\text{all } \phi^{B_0 \mathfrak{M}''(p') . B_0 \mathfrak{M}'(p'')}];$$

(4) $$\mathfrak{M}'^{D_1} . \mathfrak{M}''^{LCD_1} . \mathfrak{F} \equiv [\text{all } \phi^{B_1\mathfrak{M}'\mathfrak{M}'' . B_0\mathfrak{M}''(p')}];$$

(5) $$\mathfrak{M}'^{LCD_1} . \mathfrak{M}''^{LCD_1} . \mathfrak{F} \equiv [\text{all } \phi^{B_1\mathfrak{M}'\mathfrak{M}'' . B_0\mathfrak{M}''(p') . B_0\mathfrak{M}'(p'')}],$$

*imply the simultaneous conclusions:*

(6) $$(\mathfrak{A}\mathfrak{M}'\mathfrak{M}'')^{B_0\mathfrak{F}} . \mathfrak{F}^{LCD_1} . \mathfrak{F} = \mathfrak{F}_* . (\mathfrak{M}'\mathfrak{M}'')_*{}^{B_0\mathfrak{F}};$$

(7) $$\mathfrak{M}'^{D_0} . \mathfrak{M}''^{D_0} . \supset . \mathfrak{F}^{D_0};$$

(8) $$\mathfrak{M}'^{A} . \mathfrak{M}''^{A} . \supset . \mathfrak{F}^{A}.$$

*Further, in cases* (1, 4, 5),

(9) $$\mathfrak{F}^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''},$$

*and accordingly, in those cases,*

(10) $$\mathfrak{M}'^{D} . \mathfrak{M}''^{D} . \supset . \mathfrak{F}^{D}.$$

*Further, in case* (1),

(11) $$\theta^{B_1\mathfrak{F}} . \supset . \theta^{B_0\mathfrak{F}}.$$

As a corollary, we notice that the conclusions (6–11) hold as stated, if throughout the hypotheses (1–5) $\mathfrak{M}'$ be replaced by $\mathfrak{A}\mathfrak{M}'$ and $\mathfrak{M}''$ by $\mathfrak{A}\mathfrak{M}''$.

Besides the five cases (1–5) we take a sixth (2′) derived from case (2) by interchange of rôles of $\mathfrak{M}'$, $\mathfrak{M}''$, and we denote the class $\mathfrak{F}$ for case ($i$) by $\mathfrak{F}^i$ ($i = 1, 2, 3, 4, 5, 2'$). We readily prove directly from the specified hypotheses that $\mathfrak{F}^1$, $\mathfrak{F}^2$ have the specified properties, and then that $\mathfrak{F}^3$, $\mathfrak{F}^4$, $\mathfrak{F}^5$ have the specified properties, in view of the relations:

$$\mathfrak{F}^3 = \cap\, [\mathfrak{F}^2, \mathfrak{F}^{2'}]; \qquad \mathfrak{F}^4 = \cap\, [\mathfrak{F}^1, \mathfrak{F}^2]; \qquad \mathfrak{F}^5 = \cap\, [\mathfrak{F}^1, \mathfrak{F}^2, \mathfrak{F}^{2'}],$$

that is, the class $\mathfrak{F}^3$, defined in terms of certain classes $\mathfrak{M}'$, $\mathfrak{M}''$ satisfying the hypothesis (3), is the greatest common subclass of the classes $\mathfrak{F}^2$, $\mathfrak{F}^{2'}$ defined in terms of the same classes $\mathfrak{M}'$, $\mathfrak{M}''$; and similarly the classes $\mathfrak{F}^4$, $\mathfrak{F}^5$ are the greatest common subclasses of the classes indicated.

### *Classes on* $\mathfrak{P}^{\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III}}$ *with the property* $B_{1*}$. § 65.

65. **Theorem I.** *The classes* $\mathfrak{M}^{\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III};\,\mathrm{III}_0;\,\mathrm{III}_e}$ *have the property* $B_{1*}$ *defined in* §§ 61, 63.

t these classes $\mathfrak{M}'$ have the property $B_{1*}$ depends upon
opositions (4, 5) of § 64 and upon the two additional
erations $a$), $b$).

*The class* $\mathfrak{M}'$ *contains the developmental system* $\mathfrak{D}'$ *of*
*ons* $\delta'_k$ :

$$\delta'_{kp'} = 0 \quad (p' \neq k), \qquad 1 \quad (p' = k),$$

for the cases: I ; II$_n$ ; III (and III$_0$, III$_e$), $k$ has the
:

$$) \quad k = 1 ; \qquad \mathrm{II}_n) \quad k = 1, \cdots, n ; \qquad \mathrm{III}) \quad k = 1, 2, 3, \cdots.$$

*The class* $\mathfrak{M}'$ *has the property* $K_1'$ :

$$\mathfrak{M}'^{K_1'} . \equiv . \mathfrak{M}' \ni (\mu' .\therefore \supset .\therefore \exists \ \mu_0' \ni$$

$$e : \supset : \exists \ m_e \ni p' \geqq m_e . \supset . A\mu'_{p'} \leqq eA\mu'_{0p'}).$$

roperty $K_1'$ of classes $\mathfrak{M}'$ of functions on $\mathfrak{P}'^{\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III}}$ is the
ty $K_1'$ derived in the sense of § 61 from the bipartite
ty* $K_1'$ :

$$\phi'^{K_1'\mathfrak{M}'} . \equiv . (\phi' . \mathfrak{M}') \ni$$

$$(\exists \ \mu' \ni e : \supset : \exists \ m_e \ni p' \geqq m_e . \supset . A\phi'_{p'} \leqq eA\mu'_{p'}),$$

ich we at once define the property $K_1'\mathfrak{M}'(\mathfrak{M}'')$ of $\phi$ :

$$\exists \ (\mu'', \mu') \ni e : \supset : \exists \ m_e \ni p' \geqq m_e . \supset . A\phi_{p'} \leqq eA\mu'_{p'}\mu''.$$

notice that

$$\mathfrak{M}'^{LD_1} \ni \exists \ \mu'^{\text{nowhere zero}} : \supset : \phi'^{K_1'\mathfrak{M}'} . \supset . \phi'^{B_1\mathfrak{M}'} ;$$

$$\mathfrak{P}'^{\mathrm{III}} . \mathfrak{M}'^{B_1\mathfrak{R}\Delta'} : \supset : \phi'^{K_1'\mathfrak{M}'} . \supset . \mathbf{L}_{p'\,\infty} \phi'_{p'} = 0.$$

§ 23$e$5 we saw that the classes $\mathfrak{M}'^{\mathrm{III};\,\mathrm{III}_0;\,\mathrm{III}_e}$ on $\mathfrak{P}'^{\mathrm{III}}$ have
roperty $K_1'$. Every class $\mathfrak{M}'$ on $\mathfrak{P}'^{\mathrm{II}_n}$ has the property
aeuously, in that for every pair $\mu'$, $\mu_0'$ of functions of $\mathfrak{M}'$

---

is bipartite property $K_1'$ and the derived properties are of special
ce. Cf. §§ 23$c$, 23$d$. They are to be developed into properties of
general reference, fundamental in the theory of the $*$-composition of
of functions. Cf. §§ 67, 72, 75, 77.

the relation $A\mu'_{p'} \leqq eA\mu'_{0p'}$ holds for every $e$ and for every $p' > n$, there being no such elements $p'$ in $\mathfrak{P}'^{\mathrm{II}_n}$.

*c*) *Propositions.* —

$$(7)\qquad \begin{gathered} \mathfrak{P}'^{\mathrm{I}\,;\,\mathrm{II}_n} \,.\, \mathfrak{M}' \,.\, \mathfrak{D}'^{B_0\mathfrak{M}'} \,.\, \mathfrak{P}'' \,.\, \mathfrak{M}'' : \supset : \\ \mathfrak{F} \equiv [\text{all } \phi^{B_0\mathfrak{AM}''(p')}] \,.\, \supset \,.\, \mathfrak{F}^{B_0(\mathfrak{M}'\mathfrak{M}'')_L}\,; \end{gathered}$$

$$(8)\qquad \begin{gathered} \mathfrak{P}'^{\mathrm{I}\,;\,\mathrm{II}_n\,;\,\mathrm{III}} \,.\, \mathfrak{M}'^{K_1'} \,.\, \mathfrak{D}'^{B_0\mathfrak{M}_{\prime}} \,.\, \mathfrak{P}'' \,.\, \mathfrak{M}'' : \supset : \\ \mathfrak{F} \equiv [\text{all } \phi^{B_1\mathfrak{AM}'\mathfrak{M}'' \,.\, B_0\mathfrak{AM}''(p')}] \,.\, \supset \,.\, \mathfrak{F}^{B_0(\mathfrak{M}'\mathfrak{M}'')_*}\,. \end{gathered}$$

Here the case $\mathrm{II}_1$ is the case I, and for the case $\mathrm{II}_n$ proposition (8) is a corollary of proposition (7).

In proof of (7) for the case $\mathrm{II}_n$ and of (8) for the case III, consider a function $\phi$ of the class $\mathfrak{F}$, whether of (7) or (8), and set

$$(9)\qquad \theta_{mp'p''} \equiv \sum_{k=1}^{m} \delta'_{kp'}\phi_{kp''} \qquad \begin{pmatrix} \mathrm{II}_n : m = 1,\, 2,\, \cdots,\, n \\ \mathrm{III} : m = 1,\, 2,\, \cdots,\, n \end{pmatrix},$$

or, more briefly,

$$(10)\qquad \theta_m \equiv \sum_{k=1}^{m} \delta'_k\phi_k\,.$$

From the hypotheses :

$$\mathfrak{D}'^{B_0\mathfrak{M}'}\,;\ \phi^{B_0\mathfrak{AM}''(p')},$$

we see that

$$(11)\qquad \theta_m^{B_0(\mathfrak{M}'\mathfrak{M}'')_L} \quad (m).$$

Further, by the definition of the functions $\delta'_k$, we have

$$(12)\qquad p' \leqq m \,.\, \supset \,.\, \theta_{mp'} - \phi_{p'} = 0,$$

$$(13)\qquad p' > m \,.\, \supset \,.\, \theta_{mp'} - \phi_{p'} = -\phi_{p'},$$

so that

$$(14)\qquad m \,.\, p' \,.\, \supset \,.\, A(\theta_{mp'} - \phi_{p'}) \leqq A\phi_{p'}\,.$$

The desired conclusion of proposition (7) for the case $\mathrm{II}_n$ follows directly from (11, 12) for $m = n$.

Proceeding with the proof of proposition (8) for the case III, from the hypothesis :

$$(15)\qquad \phi^{B_1\mathfrak{AM}'\mathfrak{M}''},$$

we have, for certain $a\,;\ \mu'\,;\ \mu''$,

$$(15')\qquad A\phi \leqq Aa\mu'\mu'',$$

and hence, in view of the hypothesis:

(16) $$\mathfrak{M}'^{\,K_1'},$$

for a certain $\mu_0'$,

(17) $$e : \supset : \mathcal{H}\ p_e' \ni p' > p_e' . \supset . A\phi_{p'} \leqq eA\mu'_{0\,p'}\mu''.$$

Hence, with reference to (14), we have

(18) $$e . m . p' > p_e' . \supset . A(\theta_{mp'} - \phi_{p'}) \leqq eA\mu'_{0p'}\mu''.$$

Also, from (12), we have

(19) $$e . m \geqq p_e' . p' \leqq p_e' . \supset . \theta_{mp'} - \phi_{p'} = 0.$$

Thus, from (18, 19), we have

(20) $$e . m \geqq p_e' . p' . \supset . A(\theta_{mp'} - \phi_{p'}) \leqq eA\mu'_{0p'}\mu'',$$

or, omitting explicit mention of the now unconditioned $p'$,

(20′) $$e . m \geqq p_e' . \supset . A(\theta_m - \phi) \leqq eA\mu_0'\mu'',$$

and hence we have

(21) $$\underset{m}{\mathbf{L}}\,\theta_m = \phi \quad (\mathfrak{P}'\mathfrak{P}'';\ \mathfrak{M}'\mathfrak{M}'').$$

Accordingly, from (11, 21) we have the desired conclusion

(22) $$\phi^{B_0(\mathfrak{M}'\mathfrak{M}'')_*},$$

of proposition (8).

*d*) As a corollary of *c*) and propositions (4, 5) of § 64 w
have

THEOREM II. *In case:*

(23) $$\mathfrak{P}' \equiv \mathfrak{P}'^{\,\mathrm{I;\,II}_a\mathrm{;\,III}} . \mathfrak{M}'^{\,D_1K_1'} . \mathfrak{D}'^{\,B_0\mathfrak{M}'},$$

*it follows that, if*

(24) $$\mathfrak{P}'' . \mathfrak{M}''^{\,LCD_1} . \mathfrak{F} \equiv [\text{all } \phi^{B_1\mathfrak{M}'\mathfrak{M}''.\,B_0\mathfrak{M}''(p')}],$$

*then*

(25) $$(\mathfrak{M}'\mathfrak{M}'')_* = \mathfrak{F} . \mathfrak{F}^{LCD_1};$$

(26) $$(\mathfrak{M}'\mathfrak{M}'')^{B_0\mathfrak{F}} . \mathfrak{F}^{B_1\mathfrak{M}'\mathfrak{M}''},$$

*and, if further*

(23′; 23″) $$\mathfrak{M}'^{\,D};\ \mathfrak{M}'^{\,LC},$$

*then respectively*

(27′ ; 27″) $\mathfrak{M}''^{D} \,.\supset.\, \mathfrak{F}^{D}\,;\ \phi \,.\supset.\, \phi^{B_0\mathfrak{M}'(p'')}.$

Corollary 1. *From* (23) *it follows that*

(28) $$\mathfrak{M}'_{*} = [\text{all } \phi'^{\,B_1\mathfrak{A}\mathfrak{M}'}] \,.\, \mathfrak{M}'^{\,LCD_1A}_{*}\,;$$

(29) $$\mathfrak{M}'^{\,B_{1*}}.$$

Corollary 2. *From*

(30) $$\mathfrak{P}' \equiv \mathfrak{P}'^{\,\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III}} \,.\, \mathfrak{M}'^{\,LCD_1K_1'} \,.\, \mathfrak{D}'^{\,B_0\mathfrak{M}'}$$

*it follows that*

(31)* $$\mathfrak{M}' = [\text{all } \phi'^{\,B_1\mathfrak{M}'}] \,.\, \mathfrak{M}'^{\,B_{1*}A},$$

(32) $$\mathfrak{P}'' \,.\, \mathfrak{M}'' :\supset: \phi^{B_1\mathfrak{M}'\mathfrak{M}''} \,.\supset.\, \phi^{B_0\mathfrak{M}'(p'')}.$$

The preceding statements hold if in (23, 23′, 23″) we replace $\mathfrak{M}'$ by $\mathfrak{A}\mathfrak{M}'$.

*e*) The classes $\mathfrak{M}' \equiv \mathfrak{M}'^{\,\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III};\,\mathrm{III}_0;\,\mathrm{III}_c}$ satisfy (23, 23′, 23″, 30) and accordingly (28, 31, 32) and for them (24) implies (25, 26, 27′, 27″). Thus we have, in particular, Theorem I: *the classes* $\mathfrak{M}' \equiv \mathfrak{M}'^{\,\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III};\,\mathrm{III}_0;\,\mathrm{III}_c}$ *have the property* $B_{1*}$.

### *Classes on* $\mathfrak{P}^{\mathrm{IV}}$ *with the property* $K_{2*}$. § 66.

66. Theorem I. *The class* $\mathfrak{M}^{\mathrm{IV}}$ *of all continuous functions on a linear interval* $\mathfrak{P}^{\mathrm{IV}}$ *of the real number system has the property* $K_{2*}$ *defined in* §§ 61, 62.

This theorem for the general interval is readily deduced by transformation from the theorem for the interval (01), to the proof of which we now proceed, obtaining by the way other theorems similarly extensible.

*a*) *The development* $\Delta'$ *and its representative system.*— $\mathfrak{P}' \equiv \mathfrak{P}'^{\,\mathrm{IV}}$ denotes the interval:

(1) $$0 \leqq p' \leqq 1,$$

---

[*] ($31_1$) is expressible thus: the class $\mathfrak{M}'$ is such that the properties: $B_0\mathfrak{M}'$; $B_1\mathfrak{M}'$, of a function $\phi'$ on $\mathfrak{P}'$ are equivalent. Evidently $\phi'^{\,B_0\mathfrak{M}'}$ implies $\phi'^{\,B_1'\mathfrak{M}}$. The force of ($31_1$) is that, conversely, $\phi'^{\,B_1\mathfrak{M}'}$ implies $\phi'^{\,B_0\mathfrak{M}'}$, i. e., the class $\mathfrak{M}'$ contains every function $\phi'$ which it dominates. Of this ($31_3$) : the class $\mathfrak{M}'$ is absolute, is an immediate consequence.

of the real number system. For the system:

$$(2) \qquad ((ml)) \quad (l \leqq m\,;\ m = 1, 2, 3, \cdots;\ l = 0, 1, 2, \cdots),$$

of indices * $ml$ we introduce the corresponding systems:

$$(3) \qquad ((r'^{ml}))\,; \quad \Delta' \equiv ((\mathfrak{P}'^{ml})),$$

of rational numbers $r'^{ml}$:

$$(4)\dagger \qquad r'^{ml} \equiv \frac{l}{m}\,;$$

and of subintervals $\mathfrak{P}'^{ml}$:

$$(5) \qquad \mathfrak{P}'^{ml} \equiv [p'^{ml}], \quad r'^{m\,l-1} \leqq p'^{ml} \leqq r'^{m\,l+1},$$

where as an additional understanding:

$$(4') \qquad r'^{m\,-1} \equiv 0\,; \quad r'^{m\,m+1} \equiv 1.$$

The system:

$$(6) \qquad \Delta' \equiv ((\mathfrak{P}'^{ml})),$$

is a *development* of the class $\mathfrak{P}'$, the *stage* $m$ *of the development* being the system:

$$(7) \qquad \Delta'^{m} \equiv (\mathfrak{P}'^{ml}),$$

of $m + 1$ intervals $\mathfrak{P}'^{ml}$ $(0 \leqq l \leqq m)$.

The interval $\mathfrak{P}'^{ml}$ is of length $2/m$ $(0 < l < m)$ or $1/m$ $(l = 0, m)$, and contains the number $r'^{ml}$ which is said to *represent* it and its numbers $p'^{m}$. On stage $m$ every number $p'$ has two representative numbers $r'^{ml}$ for adjacent values of $l$; a number $r'^{ml}$ $(0 < l < m)$ however has three representative numbers, viz., besides itself both of those with adjacent values of the second part of the index.

The superscript index $ml$ used with notations $p'$ is used to denote the following property: *belonging to the interval* $\mathfrak{P}'^{ml}$ *of the development* $\Delta'$, of elements $p'$. Thus we have the notations:

$$(8) \qquad p'^{ml}\,;\ p'^{-ml},$$

the latter denoting an element $p'$ not belonging to $\mathfrak{P}'^{ml}$.

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* Throughout § 66 an index $ml$ is a pair $m$, $l$ of integers conditioned as indicated in (2).

† The superscript $ml$ is merely an index. It is not an exponent.

b) *The relation*[*] $K_2'$.—The relation[†] $K_2'$ is a property[†] on $\mathfrak{P}'\mathfrak{P}'\mathfrak{J}$, where $\mathfrak{J} \equiv [m]$, viz., a property of pairs $p_1'$, $p_2'$ of elements of $\mathfrak{P}'$ with integers $m$, having reference to the development $\Delta'$ of $\mathfrak{P}'$. We write

(9) $$K'_{p_1'p_2'm},$$

to denote that $p_1'$, $p_2'$ are two elements belonging to the same interval (say $\mathfrak{P}'^{m_0l_0}$) of some stage $m_0 \geqq m$ of the development $\Delta'$; in symbols:

(9′) $$\exists\ (m_0 \geqq m \,.\, l_0 \leqq m_0) \ni (p_1'^{\,m_0l_0} \,.\, p_2'^{\,m_0l_0}).$$

One understands how questions of infinitesimal nature on $\mathfrak{P}'$ may be treated in terms of the development $\Delta'$ of $\mathfrak{P}'$, the expression:

(10) $$\exists\ d_e \ni A(p_1'-p_2') \leqq d_e,$$

being replaced by

(11) $$\exists\ m_e \ni K'_{p_1'p_2'm_e}.$$

Thus, with reference to the development $\Delta'$ in terms of conditions $K'_{p_1'p_2'm}$ we give definitions to the properties:

(12) $$K_2';\ K_2'\mathfrak{M}';\ K_2'\mathfrak{M}'(\mathfrak{M}'');\ K_{2*}',$$

defined in § 62; the new definitions being equivalent to the former definitions. This property $K_2'$ is a bipartite property of functions and classes of functions on $\mathfrak{P}'^{\mathrm{IV}}$.

c) *Developmental systems* $\mathfrak{D}'$.—With reference to the development $\Delta'$ and its system $((r'^{ml}))$ of representative numbers a *developmental system*[‡] $\mathfrak{D}'$ is a system:

(13) $$\mathfrak{D}' \equiv [\delta'] \equiv ((\delta'^{ml})),$$

of functions on $\mathfrak{P}'^{\mathrm{IV}}$ such that

---

[*] This relation $K_2'$ has reference to the development $\Delta'$. For the general development $\Delta'$ of the general class $\mathfrak{P}'$ (cf. § 77) it is denoted by $K_2'^{\Delta'}$.

[†] The term *relation* is used for this property $K_2'$ on $\mathfrak{P}'\mathfrak{P}'\mathfrak{J}$, since we are to be concerned also with a bipartite *property* $K_2'$ of functions and classes of functions on $\mathfrak{P}'$. The property $K_2'$ is defined in terms of the relation $K_2'$.

[‡] Cf. § 78 for the general definition of developmental systems $\mathfrak{D}(\mathfrak{M})$ with respect to a development $\Delta$ and a class $\mathfrak{M}$.

(14) $$m \,.\, l \,.\, \supset .\, 0 \leqq \delta'^{\,ml} \leqq 1. \quad \delta'^{\,ml} = \begin{cases} 0 & (p'^{\,-ml}), \\ 1 & (p' \equiv r'^{\,ml}); \end{cases}$$

(15) $$m \,.\, \supset .\, \sum_{l=0}^{l=m} \delta'^{\,ml} = 1\,.$$

Hence, for every stage $m$ at an argument $p'$ which is a representative number $r'^{\,ml}$ the corresponding developmental function $\delta'^{\,ml}$ takes the value 1 and the other developmental functions $\delta'^{\,m}$ of stage $m$ vanish, while at an argument $p'$ not itself a representative number with two representatives $r'^{\,ml}$, $r'^{\,ml+1}$ the corresponding developmental functions $\delta'^{\,ml}$, $\delta'^{\,ml+1}$ take values of the interval (01) whose sum is 1, and the other developmental functions $\delta'^{\,m}$ of stage $m$ vanish.

Such a developmental system $\mathfrak{D}'$ plays a rôle in the theory of functions on $\mathfrak{P}'^{\,\mathrm{IV}}$ like the rôle of the developmental system $\mathfrak{D}'$ of § 65 in the theory of functions on $\mathfrak{P}'^{\,\mathrm{III}}$.

For the simplest developmental system † $\mathfrak{D}'$ of continuous functions the function $\delta'^{\,ml}$ has on $\mathfrak{P}'^{\,ml}$ the following specification:

(16) $$\delta'^{\,ml}_{p'} = \begin{cases} mp' - l + 1 & (r'^{\,ml-1} \leqq p' \leqq r'^{\,ml}), \\ -mp' + l + 1 & (r'^{\,ml} \leqq p' \leqq r'^{\,ml+1}). \end{cases}$$

*d*) *Developmental property* $\Delta'$. — A class $\mathfrak{M}'$ is said to have the *developmental property* $\Delta'$ in case there exists a developmental system $\mathfrak{D}'$, relative to the development $\Delta'$ and the representative system $((r'^{\,ml}))$, which belongs to the class $\mathfrak{M}'$. (Cf. § 79.)

*e*) *Propositions.* — We notice first the proposition:

(17) $$\mathfrak{D}' \,.\, \mathfrak{M}'^{\,\mathrm{IV}} \,.\, \supset .\, \mathfrak{M}'^{\,B_{\mathfrak{P}'}L,\,\mathfrak{A}\mathfrak{P}'},$$

---

† This system of continuous functions is the point of departure for Borel's determination, by the mediation of the known theorem of Weierstrass, of a system of polynomials $\pi'^{\,ml}$ such that for a continuous function $\phi'$ the sequence $\{\eta'_m\}$ of polynomials $\eta'_m$:

$$\eta'_m \equiv \sum_{l}^{m} {}_{0}\, \phi'_{r'ml}\, \pi'^{\,ml},$$

converges on $\mathfrak{P}'^{\,\mathrm{IV}}$ uniformly to $\phi'$. Cf. Borel, *Sur l'interpolation des fonctions continues par des polynomes*, Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg, 1904 (1905), pp. 224-232; and *Leçons sur les fonctions de variables réelles et les développements en séries de polynomes*, 1905, p. 80.

that is, $\mathfrak{D}'$ *being a developmental system,*[*] *every continuous function* $\mu'$ *is expressible as the limit of a uniformly convergent sequence* $\{\delta'_{L\,m}\}$ *of functions of* $\mathfrak{D}'_L$, *the linear extension of* $\mathfrak{D}'$.

In proof, let us consider a function $\phi'$ on $\mathfrak{P}'$ and set

$$(18)\qquad c_{ml} \equiv \phi'_{r'^{ml}} \quad (ml);$$

$$(19)\qquad \theta'_m \equiv \sum_{l=0}^{m} c_{ml}\,\delta'^{\,ml} \quad (m),$$

so that the sequence $\{\theta'_m\}$ belongs to $\mathfrak{D}'_L$. Then, from the definitional properties (14, 15) of $\mathfrak{D}'$, we have the relations:

$$(20)\qquad p' = r'^{\,ml} \,.\supset.\, \theta'_{mp'} = c_{ml};$$

$$(21)\qquad p' \text{ of } (r'^{\,ml}\, r'^{\,m\,l+1}) \,.\supset.\, \theta'_{mp'} \text{ of } (c_{ml}\, c_{m\,l+1}),$$

from which follows for every $p'$

$$(22)\qquad \begin{aligned} &A(\phi'_{p'} - \theta'_{mp'}) \\ &\quad \leqq A(\phi'_{p'} - \phi'_{r_1'}) + A(\phi'_{r_1'} - \theta'_{mp'}) \\ &\quad \leqq A(\phi'_{p'} - \phi'_{r_1'}) + A(\phi'_{r_1'} - \phi'_{r_2'}) \\ &\quad \leqq 2A(\phi'_{p'} - \phi'_{r_1'}) + A(\phi'_{p'} - \phi'_{r_2'}), \end{aligned}$$

where $r'_1$, $r'_2$ are two representatives $r'^{\,ml}$, $r'^{\,m\,l+1}$ of $p'$ of stage $m$. Then, if $\phi'$ is a continuous function, from (22) and the uniformity of continuity of $\phi'$ follows the desired conclusion:

$$(23)\qquad \phi' = \underset{m}{\mathrm{L}}\, \theta'_m \quad (\mathfrak{P}';\ 1),$$

of proposition (17).

It is interesting, and for our purposes essential, to analyze the proposition (17) in connection with the properties (12) associated with the bipartite property $K'_2$. The property $K'_2\mathfrak{A}(\mathfrak{P}')$ of a function $\phi'_1$ is the property of uniform continuity of $\phi'$ on the interval $\mathfrak{P}'^{\mathrm{IV}}$. Accordingly, since

$$\mathfrak{P}'^{\,\mathrm{IV}} = [\text{all } \phi'^{\,K_2'\mathfrak{A}(\mathfrak{P}')}],$$

we have as a restatement of (17) the proposition:

$$(24)\qquad \mathfrak{D}' \,.\, \mathfrak{F}' \equiv [\text{all } \phi'^{\,K_2'\mathfrak{A}(\mathfrak{P}')}] \,.\supset.\, \mathfrak{F}'^{\,B_0 \mathfrak{D}_L \mathfrak{A}(\mathfrak{P})},$$

---

[*] Whose functions $\delta'$ are continuous or discontinuous.

which is a special case of the proposition:

$$(25)\qquad \mathfrak{D}' . \mathfrak{M}' . \mathfrak{F}' \equiv [\text{all } \phi'^{K_2'\mathfrak{M}'}] . \supset . \mathfrak{F}'^{B_0\mathfrak{D}'L\mathfrak{M}'},$$

readily obtainable by the use of the last inequality of (22). Similarly we obtain the proposition:

$$(26)^* \quad \mathfrak{P}'^{IV} . \mathfrak{D}' . \mathfrak{M}' . \mathfrak{P}'' . \mathfrak{M}'' : \supset : \\ \mathfrak{F} \equiv [\text{all } \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') . B_0\mathfrak{A}\mathfrak{M}''(p')}] . \supset . \mathfrak{F}^{B_0(\mathfrak{D}'\mathfrak{M}'')L(\mathfrak{M}'\mathfrak{M}'')},$$

and its corollary:

$$(27)^* \quad \mathfrak{P}'^{IV} . \mathfrak{M}'^{\Delta'} . \mathfrak{P}'' . \mathfrak{M}'' : \supset : \\ \mathfrak{F} \equiv [\text{all } \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') . B_0\mathfrak{A}\mathfrak{M}''(p')}] . \supset . \mathfrak{F}^{B_0(\mathfrak{M}'\mathfrak{M}'')_*}.$$

*The proposition (27) is to be compared with § 65.8.*

*f) Propositions comparable with the propositions of § 64.—The respective hypotheses:*

$$(28)\quad \mathfrak{P}'^{IV} . \mathfrak{M}'^{D} . \mathfrak{P}'' . \mathfrak{M}''^{D} . \mathfrak{F} \equiv [\text{all } \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') . B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}];$$

$$(29)\quad \mathfrak{P}'^{IV} . \mathfrak{M}'^{D} . \mathfrak{P}'' . \mathfrak{M}''^{LCD} . \mathfrak{F} \equiv [\text{all } \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') . B_1\mathfrak{M}'\mathfrak{M}'' . B_0\mathfrak{M}''(p')}];$$

$$(30)\quad \mathfrak{P}'^{IV} . \mathfrak{M}'^{LCD} . \mathfrak{P}'' . \mathfrak{M}''^{LCD} . \\ \mathfrak{F} \equiv [\text{all } \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') . B_1\mathfrak{M}'\mathfrak{M}'' . B_0\mathfrak{M}''(p') . B_0\mathfrak{M}'(p'')}],$$

*imply the simultaneous conclusions:*

$$(31)\qquad \mathfrak{F}^{LC} . \mathfrak{F} = \mathfrak{F}_{\ddagger} . \mathfrak{F}^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''};$$

$$(32)\qquad \mathfrak{M}'^{K_2'} . \supset . (\mathfrak{M}'\mathfrak{M}'')_*^{B_0\delta} . \mathfrak{F}^{D}.$$

We recall the definition:

$$\mathfrak{M}'^{K_2'} : \equiv : \mathfrak{M}'^{K_2'\mathfrak{M}'} : \equiv : \mathfrak{M}' \ni \mu'^{B_0\mathfrak{M}'} . \supset . \mu'^{K_2'\mathfrak{M}'},$$

---

*26) For every developmental system $\mathfrak{D}$ and class $\mathfrak{M}'$ on $\mathfrak{P}'^{IV}$ and class $\mathfrak{M}''$ on $\mathfrak{P}''$ the class $\mathfrak{F}$ of all functions $\phi$ on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ having the properties: $K_2'\mathfrak{M}'(\mathfrak{M}'')$; $B_0\mathfrak{A}\mathfrak{M}''(p')$, is a subclass of the class obtained by extending as to $\mathfrak{M}'\mathfrak{M}''$ the class $\mathfrak{D}'\mathfrak{M}''$ extended to be linear.

27) For every class $\mathfrak{M}'$ on $\mathfrak{P}'^{IV}$, containing a developmental system $\mathfrak{D}'$, and class $\mathfrak{M}''$ on $\mathfrak{P}''$ the class $\mathfrak{F}$ of all functions $\phi$ on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ having the properties: $K_2'\mathfrak{M}'(\mathfrak{M}'')$; $B_0\mathfrak{A}\mathfrak{M}''(p')$, is a subclass of the class $(\mathfrak{M}'\mathfrak{M}'')_*$.

and accordingly remark that

$$(33) \qquad \mathfrak{P}'^{\,\mathrm{IV}} \,.\, \mathfrak{K}'^{\,K_2'\mathfrak{M}'} \,.\, \mathfrak{P}'' \,.\, \mathfrak{M}'' \,.\supset .\, (\mathfrak{K}'\mathfrak{M}'')^{K_2'\mathfrak{M}'(\mathfrak{M}'')}.$$

Then, in proof of $(32_2)$ in either case $(28\,;\,29\,;\,30)$, we notice that $\mathfrak{M}'^{K_2'}$ implies $(\mathfrak{M}'\mathfrak{M}'')^{K_2'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_{\mathfrak{M}''}}$, and hence, since $\mathfrak{M}'$ has the dominance property $D$ and $\mathfrak{K}$ is dominated by $\mathfrak{A}\mathfrak{M}'\mathfrak{M}''$, we see that $\mathfrak{K}$ has the dominance property $D$. Further, since $\mathfrak{M}'\mathfrak{M}''$ belongs to $\mathfrak{K}$ we see that $(32_1)$: $(\mathfrak{M}'\mathfrak{M}'')_L$ *belongs to* $\mathfrak{K}$, is a corollary of the fact affirmed in $(31)$ that $\mathfrak{K}$ is linear and closed. Further, $(31_2)$ is implied by $(31_1)$, and $(31_3)$ is obvious in either case $(28\,;\,29\,;\,30)$. Further $(31_1)$ for cases $(29\,;\,30)$ is implied by $(31_1)$ for case $(28)$, in view of § 64.4, 5.

We prove directly $(31_1)$ for case $(28)$. The linearity of $\mathfrak{K}$ is readily proved; we proceed to prove the closure of $\mathfrak{K}$.

With the hypothesis $(28)$ we consider a sequence $\{\phi_n\}$ of the class $\mathfrak{K}$ which converges on $\mathfrak{P} = \mathfrak{P}'\mathfrak{P}''$ uniformly as to scale $\mathfrak{K}$ to a function $\theta$. By § 64.1, 9, $\theta^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}$. We are to prove $\theta^{K_2'\mathfrak{M}'(\mathfrak{M}'')}$, or its equivalent:

$$(34) \qquad \theta^{K_2'\mathfrak{A}\mathfrak{M}'(\mathfrak{M}'')}.$$

The convergence is by hypothesis uniform to some scale $\phi$:

$$(35) \qquad \underset{n}{L}\,\phi_n = \theta \quad (\mathfrak{P}\,;\,\phi),$$

and we have for every $e$ a corresponding $n_e$. Further, $\phi$ and the $\phi_n$ have the property $K_2'\mathfrak{M}'(\mathfrak{M}'')$; accordingly we have $m_e$, $m_{ne}$, $\mu'\mu''$, $\mu_n'\mu_n''$. $\phi$ has moreover the property $B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''$, with which we have $a_0\mu_0'\mu_0''$. Since $\mathfrak{M}'^{\,D}$, $\mathfrak{M}''^{\,D}$ we may unify the scales: $\mu'\mu''$, $\mu_n'\mu_n''$ $(n)$, $a_0\mu_0'\mu_0''$ to the form: $a'a''\bar{\mu}'\bar{\mu}''$, $a_n'a_n''\bar{\mu}'\bar{\mu}''$ $(n)$, $a_0a_0'a_0''\bar{\mu}'\bar{\mu}''$, and, since $K_2'\mathfrak{A}\mathfrak{M}'(\mathfrak{M}'')$ and $K_2'\mathfrak{M}'(\mathfrak{M}'')$ are equivalent, to the form: $\mu'\mu''$, $\mu'\mu''$ $(n)$, $a\mu'\mu''$, where $a = Aa_0a_0'a_0''$ and the superscripts ( ) are omitted. This is supposed to be done in advance, the $m_e$, $m_{ne}$ referring to the unified situation.

Accordingly* we have:

$$(36) \qquad p' \,.\supset .\, A\phi_{p'} \leqq Aa\mu_{p'}'\mu'';$$

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* The following relations hold uniformly in $p''$ on $\mathfrak{P}''$.

(37) $$e \,.\, K'_{p_1'p_2'm_e} \,.\, \supset \,.\, A(\phi_{p_1'} - \phi_{p_2'}) \leqq eA\mu'_{p_1'}\mu'' ;$$

(38) $$e \,.\, n \,.\, K'_{p_1'p_2'm_{ne}} \,.\, \supset \,.\, A(\phi_{np_1'} - \phi_{np_2'}) \leqq eA\mu'_{p_1'}\mu'' ;$$

(39) $$e \,.\, n \geqq n_e \,.\, \supset \,.\, A(\phi_{np'} - \theta_{p'}) \leqq eA\phi_{p'} .$$

Now

$$A(\theta_{p_1'} - \theta_{p_2'}) \leqq A(\theta_{p_1'} - \phi_{np_1'}) + A(\phi_{np_1'} - \phi_{np_2'}) + A(\phi_{np_2'} - \theta_{p_2'}).$$

Hence, in case: $n \geqq n_e$; $K'_{p_1'p_2'm_{ne}}$, we have

$$A(\theta_{p_1'} - \theta_{p_2'}) \leqq e(A\phi_{p_1'} + A\mu'_{p_1'}\mu'' + A\phi_{p_2'}),$$

and, in case further $K'_{p_1'p_2'm_1}$,

$$A(\theta_{p_1'} - \theta_{p_2'}) \leqq e(2A\phi_{p_1'} + 2A\mu'_{p_1'}\mu'') \leqq e(2a + 2)A\mu'_{p_1'}\mu''.$$

Accordingly, taking $m_e^*$ as the greater of $m_1$ and $m_{n_e e}$, we have the relation:

(40) $$e : \supset : K'_{p_1'p_2'm_e^*} \,.\, \supset \,.\, A(\theta_{p_1'} - \theta_{p_2'}) \leqq eA(2a + 2)\mu'_{p_1'}\mu'',$$

so that, as desired,

(41) $$\theta^{K_2'\mathfrak{M}'(\mathfrak{M}'')}.$$

*g*) On comparing the results of (27, 29, 30, 31, 32) we have

Theorem II. *From*

(42) $$\mathfrak{P}'^{\,IV} \,.\, \mathfrak{M}'^{\,DK_2'\Delta'}$$

*it follows that from*

(43) $$\mathfrak{P}'' \,.\, \mathfrak{M}''^{\,LCD} \,.\, \mathfrak{F} \equiv [\text{all } \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{M}''(P')}]$$

*it follows that*

(44) $$(\mathfrak{M}'\mathfrak{M}'')_* = \mathfrak{F} \,.\, \mathfrak{F}^{LCD} ;$$

(45) $$(\mathfrak{M}'\mathfrak{M}'')^{B_0\delta} \,.\, \mathfrak{F}^{B_1\mathfrak{M}'\mathfrak{M}''} ;$$

(46) $$\phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{M}''(p')} \,.\, \supset \,.\, \phi^{B_1\mathfrak{M}'\mathfrak{M}''} ;$$

*and, if further*

(42′) $$\mathfrak{M}'^{\,LC},$$

*it follows that*

(46′) $$\phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{M}''(p')} \,.\, \supset \,.\, \phi^{B_1\mathfrak{M}'\mathfrak{M}'' \,.\, B_0\mathfrak{M}'(p'')} ;$$

Corollary 1. *From* (42) *it follows that*

(47) $$\mathfrak{M}'_* = [\text{all } \phi'^{\,K_2'\mathfrak{M}'}] \,.\, \mathfrak{M}'^{\,ALCD}_* \,.\, \mathfrak{M}'_* = \mathfrak{M}'_\# ;$$

(48) $$\mathfrak{M}'^{\,K_2'*}.$$

Corollary 2. *From*

(49) $$\mathfrak{P}'^{\,\mathrm{IV}} . \mathfrak{M}'^{\,LCDK\;\;''}$$

*it follows that*

(50) $$\mathfrak{M}' = [\text{all } \phi'^{\,K_2'\mathfrak{M}'}] . \mathfrak{M}'^{\,ALCDK_2'K_2'*}.$$

Under the hypotheses: (42; (42′); 43) on $\mathfrak{P}'$, $\mathfrak{M}'$, $\mathfrak{P}''$, $\mathfrak{M}''$ we obtain ($44_1$; 46; (46′)) by observing that $(\mathfrak{M}'\mathfrak{M}'')_*$ contains $\mathfrak{F}^{(43)} \equiv \mathfrak{F}^{(27)}$ which contains $\mathfrak{F}^{(29)}$ which (contains $\mathfrak{F}^{(30)}$ which) contains $(\mathfrak{M}'\mathfrak{M}'')_*$. ($44_1$) implies ($45_1$). (31) implies ($44_2$; $45_2$). The class: $[\text{all } \phi'^{\,K_2'\mathfrak{M}'}]$, is obviously absolute and accordingly the classes: $\mathfrak{M}'_*$ in (47) and $\mathfrak{M}'$ in (50) are absolute. Further, we notice that, the class $(\mathfrak{M}'\mathfrak{M}'')_*$ of (44) is absolute, if the class $\mathfrak{M}''$ is absolute.

On analysis of the proposition: (42, 43) implies (46), we have the following propositions:

(51) $$\mathfrak{P}'^{\,\mathrm{IV}} . \mathfrak{M}'^{\,D_1\Delta'} . \supset . \mathfrak{A}(\mathfrak{P}')^{B_1\mathfrak{A}\mathfrak{M}'};$$

(52) $$\begin{gathered}\mathfrak{P}'^{\,\mathrm{IV}} . \mathfrak{M}'^{\,D_1} . \mathfrak{A}(\mathfrak{P}')^{B_1\mathfrak{A}\mathfrak{M}'} . \mathfrak{P}'' . \mathfrak{M}''^{\,D_1} : \supset : \\ \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') \,.\, B_1\mathfrak{A}\mathfrak{M}''(p')} . \supset . \phi^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''},\end{gathered}$$

so that, since $B_0\mathfrak{A}\mathfrak{M}''(p')$ implies $B_1\mathfrak{A}\mathfrak{M}''(p')$,

(53) $$\begin{gathered}\mathfrak{P}'^{\,\mathrm{IV}} . \mathfrak{M}'^{\,D_1\Delta'} . \mathfrak{P}'' . \mathfrak{M}''^{\,D_1} : \supset : \\ \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{A}\mathfrak{M}''(p')} . \supset . \phi^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}.\end{gathered}$$

*h*) We return finally to the class $\mathfrak{M}'^{\,\mathrm{IV}}$ of all continuous functions on $\mathfrak{P}'^{\,\mathrm{IV}}$. In view of the uniform continuity of its functions $\mu'$ $\mathfrak{M}'$ has the property $K_2'$, and since $\mathfrak{M}'$ contains the developmental system $\mathfrak{D}'$ given by (16) it has the property $\Delta'$. Hence $\mathfrak{M}'$ satisfies (42, 42′, 49) and accordingly (47, 48, 50) and for it (43) implies (44, 45, 46, 46′). We have proved, in particular, Theorem I: *The class* $\mathfrak{M}'^{\,\mathrm{IV}}$ *has the property* $K_{2*}'$ defined in § 62 (cf. § 66.12).

### *Comparison of the theorems of* §§ 65, 66. § 67.

67. The notions and the propositions of §§ 65, 66 are of special reference. Thus, $\mathfrak{P}'$ is in § 65 of case I, $\mathrm{II}_n$ or III,

and in § 66 of case IV. As pointing the way to the generalizations to be given subsequently we proceed to secure formulations holding uniformly for the cases I–IV of $\mathfrak{P}'$.

*a*) For cases I–III of $\mathfrak{P}'$ we introduce the *development*:

$$(1) \qquad \Delta' \equiv ((\mathfrak{P}'^{ml})) \qquad (l \leqq m\,;\ m, l = 1, 2, 3, \cdots)$$

where the class $\mathfrak{P}'^{ml}$ consists of the single element $p'^{ml} \equiv l$, which is likewise the *representative element* $r'^{ml}$. Stage $m$: $\Delta'^{m} \equiv (\mathfrak{P}'^{ml})$, of the development consists of the $m$ elements $p' \leqq m$, said to be *developed of stage m*.

As *developmental system*:

$$(2) \qquad \mathfrak{D}' \equiv ((\delta'^{ml})),$$

we introduce the system: $\mathfrak{D}' \equiv [\delta'] \equiv \{\delta'_k\}$ of § 65 *a*), setting $\delta'^{ml} \equiv \delta'_l$. For this system § 66.14 holds, and § 66.15 holds for $p' \leqq m$, viz., for $p'$ developed of stage $m$. Further, this developmental system $\mathfrak{D}'$ is the only system $\mathfrak{D}'$ satisfying in this way § 66.14, 15. We denote by $\mathfrak{M}'^{\Delta'}$ a class $\mathfrak{M}'$ containing this system $\mathfrak{D}'$.

In § 66.12 we defined anew the properties:

$$K'_2\,;\ K'_2\mathfrak{M}'\,;\ K'_2\mathfrak{M}'(\mathfrak{M}'')\,;\ K'_{2*},$$

of § 62, by means of the condition:

$$K'_{p_1' p_2' m},$$

having reference to the development $\Delta'$ of $\mathfrak{P}'^{\mathrm{IV}}$. Similarly the properties:

$$K'_1\,;\ K'_1\mathfrak{M}'\,;\ K'_1\mathfrak{M}'(\mathfrak{M}'')\,;\ K'_{1*},$$

of § 65 *b*) are definable with reference to the development $\Delta'$ in terms of the condition:

$$K'_{p'm},$$

on an element $p'$ and an integer $m$, that at some stage $m_0 \geqq m$ the element $p'$ is undeveloped, that is, for the present development, $p' > m$.

*b*) The properties $K'_1$ and $K'_2$ were originally metrically defined in terms of the respective conditions: $p' > m$, $A(p'_1 - p'_2) \leqq d$.

The properties $K_1'$ are properties of *infinitary* nature, in that they are relations holding for large values of $p'$, viz., as $p'$ tends towards $\infty$. In case IV $p'$ is limited, and accordingly the bipartite property $K_1'$ and the derived properties: $K_1'\mathfrak{M}'(\mathfrak{M}'')$, $K_1'$, hold vacuously and generally, i. e.,

$$\phi' \,.\, \mathfrak{M}' \,.\supset.\, \phi'^{\,K_1'\mathfrak{M}'}; \quad \phi \,.\, \mathfrak{M}' \,.\, \mathfrak{M}'' \,.\supset.\, \phi^{K_1'\mathfrak{M}'(\mathfrak{M}'')}; \quad \mathfrak{M}' \,.\supset.\, \mathfrak{M}'^{\,K_1'}.$$

The properties $K_2'$ are properties of *infinitesimal* nature, in that they are* relations holding for values of $p_1'$, $p_2'$ distinct and close together, viz., as $p_1'$, $p_2'$ tend to equality. In cases I–III $p_1'$ and $p_2'$ if distinct differ at least by 1, and the properties $K_2'$ hold vacuously and generally.

The properties $K_1'$ and $K_2'$ have been defined also with respect to the developments $\Delta'$, the metrical conditions being replaced by the conditions: $K'_{p'm}$, $K'_{p_1'p_2'm}$. The foregoing remarks as to the properties $K_1'$ in case IV and the properties $K_2'$ in cases I–III may be made from this standpoint. At every stage $m$ the development $\Delta'$ of $\mathfrak{P}'^{\,\mathrm{IV}}$ is *complete*, in that every element $p'$ is an element of a class $\mathfrak{P}'^{\,ml}$ of stage $m$; at stages $m \geqq n$ the development $\Delta'$ of $\mathfrak{P}'^{\,\mathrm{II}_n}$ is complete; however, at no stage $m$ is the development $\Delta'$ of $\mathfrak{P}'^{\,\mathrm{III}}$ complete. On the other hand, at every stage $m$ the developments $\Delta'$ of $\mathfrak{P}'^{\,\mathrm{I}}$, $\mathfrak{P}'^{\,\mathrm{II}_n}$, $\mathfrak{P}'^{\,\mathrm{III}}$ are *fine*, in that every class $\mathfrak{P}'^{\,ml}$ consists of a single element, while at no stage $m$ is the development $\Delta'$ of $\mathfrak{P}'^{\,\mathrm{IV}}$ fine.

*c*) Now comparing the theorems II of §§ 65 and 66, we notice that in § 66 the properties: $K_2'$ of $\mathfrak{M}'$, $K_2'\mathfrak{M}'(\mathfrak{M}'')$ of $\phi$, enter where in § 65 enter the properties: $K_1'$ of $\mathfrak{M}'$, $B_1\mathfrak{M}'\mathfrak{M}''$ of $\phi$. To remove this discrepancy we notice, for cases I–III of $\mathfrak{P}'$, that

$$(3) \qquad \mathfrak{M}'^{\,D_1\Delta'} \,.\, \mathfrak{M}''^{\,D_1} :\supset: \phi^{K_1'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{A}\mathfrak{M}''(p')} \,.\supset.\, \phi^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''};$$

$$(4) \qquad \mathfrak{M}'^{\,K_1'} \,.\, \mathfrak{M}'' :\supset: \phi^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''} \,.\supset.\, \phi^{K_1'\mathfrak{M}'(\mathfrak{M}'')};$$

$$(5) \quad \mathfrak{M}'^{\,D_1K_1'\Delta'} \,.\, \mathfrak{M}''^{\,D_1} \,.\!\cdot\!.\supset.\!\cdot\!.\, \phi^{B_0\mathfrak{A}\mathfrak{M}''(p')} :\supset: \phi^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''} \,.\sim.\, \phi^{K_1'\mathfrak{M}'(\mathfrak{M}'')}.$$

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* Apart from relations for $p_1' = p_2'$ which hold generally.

Accordingly, in § 64.4, 5 and § 65.8, *d*) for $\mathfrak{P}'^{\,\mathrm{I-III}}$ we may in the characterization of $\phi$ replace $B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''$, $B_1\mathfrak{M}'\mathfrak{M}''$ by $\bar{K}'_1\mathfrak{M}'(\mathfrak{M}'')$, in case we impose the hypotheses: $\mathfrak{M}'^{\,D_1\bar{K}_1'\Delta'}$, $\mathfrak{M}''^{\,D_1}$. As thus modified, the propositions* for cases I–III are closely comparable with the propositions for case IV, in the former $\bar{K}'_1$ entering where in the latter $\bar{K}'_2$ enters. Then, in accordance with the remarks of *b*), we may frame propositions holding uniformly for cases I–IV by introducing throughout in place of $\bar{K}'_1$ or $\bar{K}'_2$ the notation $\bar{K}'_{12}$, denoting the composite property $\bar{K}'_1\bar{K}'_2$; $\bar{K}'_{12*}$ however denotes the property of classes $\mathfrak{M}'$ defined by means of $\bar{K}'_{12}\mathfrak{M}'(\mathfrak{M}'')$ (cf. § 61.5), a property which is quite distinct from the composite property $\bar{K}'_{1*}\bar{K}'_{2*}$.

### *Systems* $(\mathfrak{A};\ \mathfrak{P};\ \bar{K}_i;\ \mathfrak{M})$. *Relations* $\bar{K}_i$. *Composition of relations* $\bar{K}_i$. §§ 68–71.

68. *Systems* $(\mathfrak{A};\ \mathfrak{P};\ \bar{K}_1,\ \bar{K}_2;\ \mathfrak{M})$.— In the first section† (§§ 51–55) of Part II we have defined the composition of classes of elements and of classes of functions. In the second section (§§ 56–67) we have obtained functional characterizations of the $*$-composite $(\mathfrak{M}'\mathfrak{M}'')_*$ of the classes $\mathfrak{P}' = \mathfrak{P}'^{\,\mathrm{I-IV}}$ with arbitrary classes $\mathfrak{M}''^{\,LCD}$; these characterizations are of the type $B_*$ of § 61 and more closely of the type $\bar{K}'_{12*}$ of § 67, related to certain developments $\Delta'$ of $\mathfrak{P}'^{\,\mathrm{I-IV}}$ by the mediation of certain relations $\bar{K}'_1$, $\bar{K}'_2$ defined in terms of the developments $\Delta'$. Before taking up in the fourth section (§§ 75–84) the study of the general development $\Delta'$ of the general class $\mathfrak{P}'$, we consider in this third section (§§ 68–74) the general relations $\bar{K}_1$, $\bar{K}_2$ of the general class $\mathfrak{P}'$.

69. *The relations* $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_{12}$.— For a class $\mathfrak{P}$ of elements a

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[*] We notice the theorem:

$$\mathfrak{P}'^{\,\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III}}\,.\,\mathfrak{M}'^{\,D_1\bar{K}_1'\Delta'}\,.\supset.\,\mathfrak{M}'^{\,\bar{K}_1'*},$$

with its corollary,

$$\mathfrak{M}'^{\,\mathrm{I};\,\mathrm{II}_n;\,\mathrm{III};\,\mathrm{III}_0;\,\mathrm{III}_c}\,.\supset.\,\mathfrak{M}'^{\,\bar{K}_1'*}.$$

[†] Cf. §§ 50, 56.

relation $K_1$ is a relation[*] on $\mathfrak{P}\mathfrak{J}$ and a relation $K_2$ is a relation on $\mathfrak{P}\mathfrak{P}\mathfrak{J}$. Here $\mathfrak{J}$ denotes the class $[m]$ of positive integers. Accordingly the notations:

$$\mathit{ad}\ K_1 : K_{pm}; \qquad \mathit{ad}\ K_2 : K_{p_1p_2m},$$

denote $pm$; $p_1p_2m$ fulfilling certain respective conditions. Thus for $\mathfrak{P}^{\text{I-IV}}$ we had the metrical instances:

$$K_{pm} \,.\equiv .\, p > m; \qquad K_{p_1p_2m} \,.\equiv .\, A(p_1 - p_2) \leqq \frac{1}{m}.$$[†]

We frequently consider simultaneously a relation $K_1$ and a relation $K_2$ for a class $\mathfrak{P}$; in such cases we speak of the relation $K_{12}$. Thus we consider relations $K_i$ ($i = 1, 2, 12$), as we may say, on $\mathfrak{P}$. At present we consider individual relations $K_1$ or $K_2$ or $K_{12}$ on $\mathfrak{P}$, postponing the consideration of classes of relations on $\mathfrak{P}$.

70. *Properties of relations* $K_1$, $K_2$, $K_{12}$.—In systems ($\mathfrak{A}$; $\mathfrak{P}$; $K_1$, $K_2$; $\mathfrak{M}$) the relations $K_1$, $K_2$ are used to define properties of classes $\mathfrak{M}$. They are of the type specified; other restrictions on generality will be made as needed. We notice the extreme instances, in which the conditions $K_{pm}$, $K_{p_1p_2m}$ are (o) never, (□) always fulfilled, viz.,

(o) $\quad {}^{-}\mathcal{E}\ (p, m) \ni K_{pm}; \qquad {}^{-}\mathcal{E}\ (p_1, p_2, m) \ni K_{p_1p_2m},$

(□) $\quad p \,.\, m \,.\supset .\, K_{pm}; \qquad p_1 \,.\, p_2 \,.\, m \,.\supset .\, K_{p_1p_2m}.$

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[*] These *relations* $K_1$, $K_2$ are properties $K_1$, $K_2$ on $\mathfrak{P}\mathfrak{J}$, $\mathfrak{P}\mathfrak{P}\mathfrak{J}$, where $\mathfrak{J}$ denotes the class $[m]$ of positive integers, in terms of which are defined in § 72 the bipartite *properties* $K_1$, $K_2$ of functions and classes of functions on $\mathfrak{P}$, and their associated properties.

[†] More generally, if $\delta$ is a nowhere negative function on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$, a function of the general nature of *distance*, e. g., FRÉCHET's *voisinage* or *écart*, an instance is

$$K_{p_1p_2m} \,.\equiv .\, \delta_{p_1p_2} \leqq \frac{1}{m}.$$

A relation $K_2$ gives rise to numerous relations $K_1$, e. g., for every element $p_0$ of $\mathfrak{P}$ there is a relation $K_1$, viz.,

$$K_{pm} \,.\equiv .\, K_{pp_0m} \,.$$

Relations $\bar{K}_1$, $\bar{K}_2$ determine uniquely and are uniquely determinable from the respective general systems:

$$(\mathfrak{P}_m);\ (\mathfrak{P}_{p_2m}),$$

of subclasses of $\mathfrak{P}$ such that

$$\bar{K}_{pm} \cdot \sim \cdot p^{B_0\mathfrak{P}_m}; \qquad \bar{K}_{p_1p_2m} \cdot \sim \cdot p_1{}^{B_0\mathfrak{P}_{p_2m}}.$$

The subclasses may be null-classes. A non-symmetric relation $\bar{K}_2$ is similarly related to a second system:

$$({}_{p_1}\mathfrak{P}_m),$$

such that

$$\bar{K}_{p_1p_2m} \cdot \sim \cdot p_2{}^{B_0\,{}_{p_1}\mathfrak{P}_m}.$$

Relations $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_{12}$ may have, for instance, the following properties:

(1) $m_0 \geqq m \cdot \bar{K}_{pm} \cdot \supset \cdot \bar{K}_{pm_0}$; $\quad m_0 \geqq m \cdot \bar{K}_{p_1p_2m} \cdot \supset \cdot \bar{K}_{p_1p_2m_0}$.

(2)[*] $\quad p \cdot m \cdot \supset \cdot \bar{K}_{pm} \smallsmile \bar{K}_{ppm}$.

(3) $\quad p \cdot m \cdot \supset \cdot \bar{K}_{ppm}$.

(4) $\quad \bar{K}_{p_1p_2m} \cdot \supset \cdot \bar{K}_{p_2p_1m}$.

(5) $\quad \bar{K}_{p_1p_2m} \cdot \supset \cdot p_1 = p_2$.

We denote relations having these respective properties as follows:

(1) $\bar{K}_1{}^1$; $\bar{K}_2{}^1$; $\bar{K}_{12}{}^1$. (2) $\bar{K}_{12}{}^2$. (3) $\bar{K}_2{}^3$. (4) $\bar{K}_2{}^4$. (5) $\bar{K}_2{}^5$.

Obviously, $(\bar{K}_1 \cdot \bar{K}_2{}^3)$ implies $\bar{K}_{12}{}^2$.

The principal notions and propositions of the sequel relate to relations $\bar{K}_{12}{}^{12}$, that is, to relations $\bar{K}_{12}$ having the properties 1, 2. In particular, the relation $\bar{K}_{12}{}^{\Delta}$, derived from a development $\Delta$ (cf. § 77) of $P$, has the properties 1, 2, 4.

71. *Composition of relations* $\bar{K}_i$. — Relations $\bar{K}_i'$ on $\mathfrak{P}'$, $\bar{K}_i''$ on $\mathfrak{P}''$, $\bar{K}_i$ on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ may be related in one or more of the following ways:

(1)
$$\bar{K}'_{p'm} \cdot p'' \cdot \supset \cdot \bar{K}_{p'p''m};$$
$$p' \cdot \bar{K}''_{p''m} \cdot \supset \cdot \bar{K}_{p'p''m}.$$

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[*] 2) Every element $p$ and integer $m$ satisfy either the condition $\bar{K}_{pm}$ or the condition $\bar{K}_{ppm}$. The symbol $\smallsmile$ denotes *or*.

$$(2)\qquad K_{p'p''m} \,.\supset .\, K'_{p'm} \smile K''_{p''m} \,.$$

$$(3)\qquad K'_{p_1'p_2'm} \,.\, K''_{p_1''p_2''m} \,.\supset .\, K_{p_1'p_1''p_2'p_2''m} \,.$$

$$(4)\qquad K_{p_1'p_1''p_2'p_2''m} \,.\supset .\, K'_{p_1'p_2'm} \,.\, K''_{p_1''p_2''m} \,.$$

$$(5)\qquad \begin{aligned} &K'_{p_1'p_2'm} \,.\, p'' \,.\supset .\, K_{p_1'p''p_2'p''m} \smile K_{p_1'p''m} K_{p_2'p''m} \,; \\ &p' \,.\, K''_{p_1''p_2''m} \,.\supset .\, K_{p'p_1''p'p''_2m} \smile K_{p'p_1''m} K_{p'p_2''m} \,. \end{aligned}$$

We denote relations $K'_i$, etc., so related as follows :

$$(1)\qquad (K'_1, K''_1, K_1)^1 ; \qquad (2)\quad (K''_1, K''_1, K_1)^2;$$

$$(3)\qquad (K'_2, K''_2, K_2)^3 ; \qquad (4)\quad (K'_2, K''_2, K_2)^4 ;$$

$$(5)\qquad (K'_2, K_{12})^5, (K''_2, K_{12})^5, (K'_2, K''_2, K_{12})^5 ;$$

or in other ways of obvious interpretation.

For given relations $K'_i$, $K''_i$ $(i = 1 ; 2 ; 12)$ there is precisely one relation $K_i$, the *composite relation*, in notation, $K'_i K''_i$ or $K'_i{}''$, satisfying the conditions :

$$(i = 1)\quad (K'_1, K''_1, K_1)^{12}; \qquad (i = 2)\quad (K'_2, K''_2, K_2)^{34};$$

$$(i = 12)\quad (K''_{12}, K''_{12}, K_{12})^{1234}.$$

The propositions of §§ 72–4 having reference to systems $(K'_i, K''_i, K_i)$ are available, as indicated in the theorems of § 74, for systems $(K'_i, K''_i, K'_i{}'')$, and also later on for systems of relations derived from developments $\Delta$, the composition of developments inducing for those relations a certain composition. In this composition of relations the composite relation $K_1$ is $K'_1{}''$, while the composite relation $K_2$ is in general distinct from $K'_2{}''$, the relation $(K'_2, K''_2, K_2)^4$ holding but the relation $(K'_2, K''_2, K_2)^3$ in general not holding. Thus this composition of relations satisfies the conditions :

$$(i = 1)\quad (K'_1, K''_1, K_1)^{12}; \qquad (i = 2)\quad (K'_2, K''_2, K_2)^4;$$

$$(i = 12)\quad (K'_{12}, K''_{12}, K_{12})^{124}.$$

We do not now consider more closely the interrelations of these two types of composition of relations derived from developments.

71*a*. *Propositions concerning the composite relations $\check{K}_i^{\prime\,\prime\prime}$ in connection with the properties* 1–5 (§ 70) *of relations.*—

$$(1)\qquad \check{K}'^{\,2}_{12} \,.\, \check{K}''_{12} \,.\supset.\, (\check{K}''_{2},\ \check{K}^{\prime\,\prime\prime}_{12})^3.$$

$$\check{K}'_{12} \,.\, \check{K}''^{\,2}_{12} \,.\supset.\, (\check{K}'_{2},\ \check{K}^{\prime\,\prime\prime}_{12})^3.$$

$$(2)\qquad \check{K}'^{\,2}_{12} \,.\, \check{K}''^{\,2}_{12} \,.\supset.\, (\check{K}'_{2},\ \check{K}''_{2},\ \check{K}^{\prime\,\prime\prime}_{12})^3.$$

(3) *The properties* 1–5 *of relations $\check{K}_i$ are invariant under composition of relations*, viz.,

$$(3_1)\qquad \check{K}'^{\,1}_{i} \,.\, \check{K}''^{\,1}_{i} \,.\supset.\, \check{K}^{\prime\,\prime\prime\,1}_{i} \qquad (i=1,2,12);$$

$$(3_2)\qquad \check{K}'^{\,2}_{12} \,.\, \check{K}''^{\,2}_{12} \,.\supset.\, \check{K}^{\prime\,\prime\prime\,2}_{12};$$

$$(3_g)\qquad \check{K}'^{\,g}_{2} \,.\, \check{K}''^{\,g}_{2} \,.\supset.\, \check{K}^{\prime\,\prime\prime\,g}_{2} \qquad (g=3,4,5).$$

## *Properties $\check{K}_i$, $\check{K}_{i*}$ of classes $\mathfrak{M}$. Composition theory of classes with the properties $D\check{K}_i$, $D\check{K}_{i*}$.* §§ 72–74.

72. *The properties $\check{K}_1$, $\check{K}_2$, $\check{K}_{12}$, etc., relative to relations $\check{K}_1$, $\check{K}_2$.*—Using the notations ′, ″ as usual, relative to relations: $\check{K}_1$, $\check{K}_2$ on $\mathfrak{P}$; $\check{K}'_1$, $\check{K}'_2$ on $\mathfrak{P}'$, and classes of functions: $\mathfrak{M}$ on $\mathfrak{P}$; $\mathfrak{M}'$ on $\mathfrak{P}'$; $\mathfrak{M}''$ on $\mathfrak{P}''$, we define properties: $\check{K}_1$; $\check{K}_2$; $\check{K}_{12}$, etc., generalizations of the properties of §§ 65 and 66, as follows:

$$(1)\quad \phi^{\check{K}_1\mathfrak{M}} \,.\equiv.\, \phi \ni (\exists\ \mu \ni e :\supset: \exists\ m_e \ni \check{K}_{pm_e} \,.\supset.\, A\phi_p \leqq eA\mu_p).$$

$$(2)\quad \phi^{\check{K}_2\mathfrak{M}} \,.\equiv.\, \phi \ni (\exists\ \mu \ni e :\supset: \exists\ m_e \ni \check{K}_{p_1p_2m_e} \,.\supset.\, A(\phi_{p_1}-\phi_{p_2}) \leqq eA\mu_{p_1}).$$

$$(3)\quad \phi^{\check{K}_{1'}\mathfrak{M}'(\mathfrak{M}'')} \,.\equiv.\, \phi \ni (\exists\,(\mu'',\mu') \ni e :\supset: \exists\ m_e \ni \check{K}'_{p'm_e} \,.\supset.\, A\phi_{p'} \leqq eA\mu'_{p'}\mu'').$$

$$(4)\quad \phi^{\check{K}_{2'}\mathfrak{M}'(\mathfrak{M}'')} \,.\equiv.\, \phi \ni (\exists\,(\mu'',\mu') \ni e :\supset: \exists\ m_e \ni \check{K}'_{p_1'p_2'm_e} \,.\supset.\, A(\phi_{p_1'}-\phi_{p_2'}) \leqq eA\mu'_{p_1'}\mu'').$$

$$(5)\qquad \phi^{\check{K}_{12}\mathfrak{M}} \,.\equiv.\, \phi^{\check{K}_1\mathfrak{M}\,.\,\check{K}_2\mathfrak{M}}.$$

$$(6)\qquad \phi^{\check{K}_{12'}\mathfrak{M}'(\mathfrak{M}'')} \,.\equiv.\, \phi^{\check{K}_{1'}\mathfrak{M}'(\mathfrak{M}'')\,.\,\check{K}_{2'}\mathfrak{M}'(\mathfrak{M}'')}.$$

Here $\phi$ is a function on $\mathfrak{P}$ to $\mathfrak{A}$ and in (3, 4, 6) $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$. Further in (1, 2, 5) $\check{K}_1$, $\check{K}_2$, $\check{K}_{12}$ are bipartite properties (cf. § 60) of functions and classes of functions on $\mathfrak{P}$, properties relative to the relations $K_1$, $K_2$ on $\mathfrak{P}$, in terms of which they are defined; the property $K_{12}$ being the composite of the properties $K_1$, $K_2$. Similarly in (3, 4, 6) the notations $K'_1$, $K'_2$, $K'_{12}$ relate to properties and relations on $\mathfrak{P}'$.

The properties $K_1$ are in a sense *infinitary* properties and the properties $K_2$ are in a sense *infinitesimal* properties, viz., with respect to the relations $K_1$, $K_2$ and the class $\mathfrak{M}$.

We notice below the extreme instances of the relations $\phi K_1\mathfrak{M}$, $\phi K_2\mathfrak{M}$.*

Other notations are introduced in accordance with §§ 57–61. For example, cf. § 61.2, 3, 5:

(7) $$\mathfrak{M}^{K_1} :\equiv: \mathfrak{M}^{K_1\mathfrak{M}} :\equiv: \mu \,.\supset .\, \mu^{K_1\mathfrak{M}}.$$

(8) $$\mathfrak{M}^{K_2} :\equiv: \mathfrak{M}^{K_2\mathfrak{M}} :\equiv: \mu \,.\supset .\, \mu^{K_2\mathfrak{M}}.$$

(9) $$\mathfrak{M}^{K_{12}} :\equiv: \mathfrak{M}^{K_{12}\mathfrak{M}} :\equiv: \mu \,.\supset .\, \mu^{K_{12}\mathfrak{M}} :\sim: \mathfrak{M}^{K_1K_2}.$$

(10) $$\mathfrak{M}'^{K_1'*} :\equiv: \mathfrak{M}' \ni \\ \mathfrak{M}''^{LCD} \,. \quad .\, (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{K_1'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}]$$

(11) $$\mathfrak{M}'^{K_2'*} :\equiv: \mathfrak{M}' \ni \\ \mathfrak{M}''^{LCD} \,.\supset .\, (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}].$$

(12)† $$\mathfrak{M}'^{K_{12}'*} :\equiv: \mathfrak{M}' \ni \\ \mathfrak{M}''^{LCD} \,.\supset .\, (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{K_{12}'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}].$$

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* For the relation $K_1 \equiv K_1^\circ$ every class $\mathfrak{M}$ and function $\phi$ are in the bipartite relation $K_1$. For the relation $K_2 \equiv K_2^\circ$ every class $\mathfrak{M}$ and function $\phi$ are in the bipartite relation $K_2$. The relations hold vacuously.

For the relation $K_1 \equiv K_1^\square$ for every class $\mathfrak{M}$ $\phi K_1\mathfrak{M}$ implies $\phi = 0(\mathfrak{P})$. For the relation $K_2 \equiv K_2^\square$ for every class $\mathfrak{M}$ $\phi K_2\mathfrak{M}$ implies that $\phi$ is of the class $\mathfrak{A}(\mathfrak{P})$.

The function $\phi \equiv 0(\mathfrak{P})$ has for every relation $K_1$ and class $\mathfrak{M}$ the property $K_1\mathfrak{M}$; and it is the only such function. The functions $\phi \equiv a(\mathfrak{P})$ have for every relation $K_2$ and class $\mathfrak{M}$ the property $K_2\mathfrak{M}$; and there are no other such functions.

† It is to be noticed that the property $K_{12}'{}_*$ of classes $\mathfrak{M}'$ of functions is not the composite of the properties $K_1'{}_*$, $K_2'{}_*$. For

$$\mathfrak{M}'^{K_{1*}'\,.\,K_{2*}'} .\equiv. \mathfrak{M}' \ni (\mathfrak{M}''^{LCD} \,.\supset .\, (\mathfrak{M}'\mathfrak{M}'')_* = [\text{all } \phi^{K_1'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{M}''(p')}] \\ = [\text{all } \phi^{K_2'\mathfrak{M}'(\mathfrak{M}'').B_0\mathfrak{M}''(p')}]).$$

72*a*. *Propositions concerning relations and classes of functions on* $\mathfrak{P}$.—The following propositions are for $i = 1 ; 2 ; 12$.

$(1_i)$ $$K_i \,.\, \mathfrak{F}^{K_i\mathfrak{M}} \,.\supset.\, (A\mathfrak{F})^{K_i\mathfrak{M}} \,.\, (\mathfrak{A}\mathfrak{F})^{K_i\mathfrak{M}}.$$

$(2_i)$ $$K_i \,.\, \mathfrak{M}_1^{B_0\mathfrak{M}_2} \,.\, \mathfrak{F}^{K_i\mathfrak{M}_1} \,.\supset.\, \mathfrak{F}^{K_i\mathfrak{M}_2}.$$

$(3_i)$ $$K_i \,.\, \mathfrak{M}_1^{B_1\mathfrak{A}\mathfrak{M}_2} \,.\, \mathfrak{F}^{K_i\mathfrak{M}_1} \,.\supset.\, \mathfrak{F}^{K_i\mathfrak{M}_2}.$$

$(3_1')$ $$K_1 \,.\, \mathfrak{F}_1^{K_1\mathfrak{M}} \,.\, \mathfrak{F}_2^{B_1\mathfrak{A}\mathfrak{F}_1} \,.\supset.\, \mathfrak{F}_2^{K_1\mathfrak{M}}.$$

$(4_i)$ $$K_i^1 \,.\, \mathfrak{M}^{D_1} \,.\, \mathfrak{F}^{K_i\mathfrak{M}} \,.\supset.\, \mathfrak{F}_L^{K_i\mathfrak{M}}.$$

$(5_i)$ $$K_i^1 \,.\, \mathfrak{M}^{D} \,.\, \mathfrak{F}_1^{K_i\mathfrak{M}} \,.\, \mathfrak{F}_2^{K_i\mathfrak{M}\,.\,B_1\mathfrak{A}\mathfrak{M}} \,.\supset.\, \mathfrak{F}_{1\mathfrak{F}_2}^{K_i\mathfrak{M}}.$$

$(6_i)$ $$K_i^1 \,.\, \mathfrak{M}^{D} \,.\, \mathfrak{F}^{K_i\mathfrak{M}\,.\,B_1\mathfrak{A}\mathfrak{M}} \,.\supset.\, \mathfrak{F}_L^{K_i\mathfrak{M}\,.\,B_1\mathfrak{A}\mathfrak{M}} \,.\, \mathfrak{F}_{\ddagger}^{K_i\mathfrak{M}\,.\,B_1\mathfrak{A}\mathfrak{M}}.$$

$(7_i)$ $$K_i^1 \,.\, \mathfrak{M}^{DK_i} \,.\supset.\, \mathfrak{M}_L^{DK_i} \,.\, \mathfrak{M}_{\mathfrak{m}}^{DK_i} \,.\, \mathfrak{M}_{\ddagger}^{DK_i}.$$

$(8_i)$[*] $$K_i^1 \,.\, \mathfrak{M} \,.\, \mathfrak{F}_i = [\text{all } \phi^{K_i\mathfrak{M}}] :\supset: \mathfrak{F}_i^{K_i\mathfrak{M}\,.\,A} : \mathfrak{F}_i = \mathfrak{A}\mathfrak{F}_i :$$
$$\mathfrak{M}^{D_1} \,.\supset.\, \mathfrak{F}_i^{L} : \mathfrak{M}^{D} \,.\supset.\, \mathfrak{F}_i^{LC} : \mathfrak{M}^{K_i} \,.\supset.\, \mathfrak{M}^{B_0\mathfrak{F}_i} \,.\, \mathfrak{F}_i^{K_i}.$$

$(8_1')$[*] $$K_1^1 \,.\, \mathfrak{M} \,.\, \mathfrak{F}_1 = [\text{all } \phi^{K_1\mathfrak{M}}] \,.\supset.\, B_0\mathfrak{F}_1 \sim B_1\mathfrak{F}_1.$$

$(9_i)$ $$K_i^1 \,.\, \mathfrak{M}^{D} :\supset: (\mathfrak{M}^{K_i} \,.\sim.\, \mathfrak{M}_{;}^{K_i}) \,.\, (K_i\mathfrak{M} \sim K_i\mathfrak{M}_{\ddagger}).$$

$(10_i)$ $$K_i^1 \,.\, \mathfrak{F}^{K_i\mathfrak{M}_{\mathfrak{F}}} \,.\sim.\, \mathfrak{F}^{K_i\mathfrak{M}}.$$

$(11_i)$ $$K_i^1 \,.\, \mathfrak{M}^{K_{i*}} \,.\supset.\, \mathfrak{M}_{|} = [\text{all } \phi^{K_i\mathfrak{M}}].$$
$$\mathfrak{M}_{|}^{K_i\mathfrak{M}_{|}A} \,.\, \mathfrak{M}^{K_i} \,.\, (B_0\mathfrak{M}_{\ddagger} \sim K_i\mathfrak{M}).$$

$(12_i)$ $$K_i^1 \,.\, \mathfrak{M}^{DK_{i*}} \,.\supset.\, \mathfrak{M}^{K_i} \,.\, \mathfrak{M}_L^{LDK_iK_{i*}\,.\,B_1\mathfrak{A}\mathfrak{M}} \,.\, \mathfrak{M}_{\ddagger}^{LCDAK_iK_{i*}\,.\,B_1\mathfrak{A}\mathfrak{M}}.$$

$(13_i)$ $$K_i^1 \,.\, \mathfrak{M}^{D} :\supset: \mathfrak{M}^{K_{i*}} \,.\sim.\, \mathfrak{M}_L^{K_{i*}} \,.\sim.\, \mathfrak{M}_{;}^{K_{i*}}.$$

$(14_i)$ $$K_i^1 \,.\, \mathfrak{M}^{LCK_{i*}} \,.\supset.\, \mathfrak{M}^{K_iA} \,.\, (B_0\mathfrak{M} \sim K_i\mathfrak{M}).$$

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[*] $\mathfrak{M}$ being a class of functions on $\mathfrak{P}$, and $K_i (i = 1; 2; 12)$ being a relation $K_i$ on $\mathfrak{P}$ having the property 1 of § 70, and $\mathfrak{F}_i$ being the class of all functions $\phi$ on $\mathfrak{P}$ having the property $K_i\mathfrak{M}$; the class $\mathfrak{F}_i$ is absolute and has the property $K_i\mathfrak{M}$ and is closed under multiplication by the class $\mathfrak{A}$ of constants; and further, if the class $\mathfrak{M}$ has the dominance property $D_1$, the class $\mathfrak{F}_i$ is linear; if the class $\mathfrak{M}$ has the dominance property $D$, the class $\mathfrak{F}_i$ is linear and closed; and if the class $\mathfrak{M}$ has the property $K_i$, the class $\mathfrak{M}$ belongs to the class $\mathfrak{F}_i$ and the class $\mathfrak{F}_i$ has the property $K_i$. Further, in the case $i = 1$, the properties: $B_0\mathfrak{F}_1$; $B_1\mathfrak{F}_1$, viz., the properties: *belonging to the class* $\mathfrak{F}_1$; *dominated by the class* $\mathfrak{F}_1$, whether of functions or of classes of functions on $\mathfrak{P}$, are equivalent.

72*b*. *Propositions concerning relations and classes of functions on* $\mathfrak{P}'$, $\mathfrak{P}''$, $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$. — The following propositions are for $i = 1;\ 2;\ 12$.

$$K'_i\,.\,\mathfrak{F}'^{K_i'\mathfrak{M}'}\,.\,\mathfrak{M}''\,.\supset.\,(\mathfrak{F}'\mathfrak{M}'')^{K_i'\mathfrak{M}'(\mathfrak{M}'')}. \tag{$1_i$}$$

$$(K_i'^{\,1}\,.\,\mathfrak{M}'^{D_1}\,.\,\mathfrak{F}'^{K_i'\mathfrak{M}'\,.\,B_1\mathfrak{A}\mathfrak{M}'})\,.\,(K_i''^{\,1}\,.\,\mathfrak{M}''^{D_1}\,.\,\mathfrak{F}''^{K_i''\mathfrak{M}''\,.\,B_1\mathfrak{A}\mathfrak{M}''}) \tag{$2_i$}$$
$$:\supset:(2_i'\,.\,2_i''),\text{ viz.,}$$

$$(K_1', K_1'', K_1)^2\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1}\,.\,(\mathfrak{F}'\mathfrak{F}'')^{K_1\mathfrak{M}'\mathfrak{M}''\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}; \tag{$2_1'$}$$

$$(K_2', K_2'', K_2)^4\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1}\,.\,(\mathfrak{F}'\mathfrak{F}'')^{K_2\mathfrak{M}'\mathfrak{M}''\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}; \tag{$2_2'$}$$

$$(K_{12}', K_{12}'', K_{12})^{24}\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1}\,.\,(\mathfrak{F}'\mathfrak{F}'')^{K_{12}\mathfrak{M}'\mathfrak{M}''\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}; \tag{$2_{12}'$}$$

$$K'''^{\,1}\,.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1}\,.\,(\mathfrak{F}'\mathfrak{F}'')^{K_{12}'''\mathfrak{M}'\mathfrak{M}''\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}. \tag{$2_i''$}$$

$$(K_i'^{\,1}\,.\,\mathfrak{M}'^{D_1K_i'})\,.\,(K_i''^{\,1}\,.\,\mathfrak{M}''^{D_1K_i''}):\supset:(3_i'\,.\,3_i''),\text{ viz.,} \tag{$3_i$}$$

$$(K_1', K_1'', K_1)^2\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1K_1}; \tag{$3_1'$}$$

$$(K_2', K_2'', K_2)^4\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1K_2}; \tag{$3_2'$}$$

$$(K_{12}', K_{12}'', K_{12})^{24}\,.\supset.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1K_{12}}; \tag{$3_{12}'$}$$

$$K_i'''^{\,1}\,.\,(\mathfrak{M}'\mathfrak{M}'')^{D_1K_{12}'''}. \tag{$3_i''$}$$

$(4_i)$ The propositions $(1_i–6_i, 8_i, 8_1')$ of § 72*a* may be generalized by the introduction of relativity. E. g., the generalization of $(6_i)$ is:

$$K_i'^{\,1}\mathfrak{M}'^{D}\,.\,\mathfrak{M}''^{D}\,.\,\mathfrak{F}^{K_i'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}\,.\supset.\,\mathfrak{F}_L{}^{K_i'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}\,.$$
$$\mathfrak{F}_*{}^{K_i'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}.$$

The general propositions concern classes $\mathfrak{M}'$ on $\mathfrak{P}'$, $\mathfrak{M}''$ on $\mathfrak{P}''$, $\mathfrak{F}$ on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$, and relations $K_i'$ on $\mathfrak{P}'$. In view of § $72a10_i$, the original propositions are for the special case: $\mathfrak{P}'' = \mathfrak{P}''^{\mathrm{I}}$; $\mathfrak{P} \equiv \mathfrak{P}'$; $\mathfrak{M}'' \equiv \mathfrak{A}$.

73. *Classes containing* $(\mathfrak{M}'\mathfrak{M}'')_*$. — The following theorem, readily deducible from preceding propositions (§ 64.1, 2; § 72*b*4), embraces (for each index $i = 1;\ 2;\ 12$) three propositions comparable with the propositions of § 64.3, 4, 5 and § 66 *d*). $K_i'^{\,1}$ denotes a relation $K_1'$ or $K_2'$ or $K_{12}'$ on $\mathfrak{P}'$ having the property 1 of § 70.

THEOREM. *The respective hypotheses:*

(1$_i$) $\bar{K}_i'^{\,1} \,.\, \mathfrak{M}'^{\,D} \,.\, \mathfrak{M}''^{\,D} \,.\, \mathfrak{F} \equiv [\text{all } \phi^{K_i'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''}]$;

(2$_i$) $\bar{K}_i'^{\,1} \,.\, \mathfrak{M}'^{\,D} \,.\, \mathfrak{M}''^{\,LCD} \,.\, \mathfrak{F} \equiv [\text{all } \phi^{K_i'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_1\mathfrak{M}'\mathfrak{M}''\,.\,B_0\mathfrak{M}''(p)}]$;

(3$_i$) $\bar{K}_i'^{\,1} \,.\, \mathfrak{M}'^{\,LCD} \,.\, \mathfrak{M}''^{\,LCD} \,.\, \mathfrak{F} \equiv [\text{all } \phi^{K_i'\mathfrak{M}'(\mathfrak{M}'')\,.\,B_1\mathfrak{M}'\mathfrak{M}''\,.\,B_0\mathfrak{M}''(p')\,.\,B_0\mathfrak{M}'(p'')}]$,

*imply the simultaneous conclusions:*

(4$_i$) $\mathfrak{F}^{\,LC\,.\,B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''} \,.\, \mathfrak{F} = \mathfrak{F}_*$;

(5$_i$) $\mathfrak{M}'^{\,K_i'} \,.\, \supset \,.\, (\mathfrak{M}'\mathfrak{M}'')_*^{\,B_0\delta} \,.\, \mathfrak{F}^{\,D}$;

(6$_i$) $\mathfrak{M}'^{\,D_0K_i'} \,.\, \mathfrak{M}''^{\,D_0} \,.\, \supset \,.\, \mathfrak{F}^{\,D_0}$,

*and, in the respective cases* (1$_i$; 2$_i$; 3$_i$),

(7$_i$) $\mathfrak{F}^{\,A}$; $\mathfrak{M}''^{\,A} \,.\, \supset \,.\, \mathfrak{F}^{\,A}$; $\mathfrak{M}'^{\,A} \,.\, \mathfrak{M}''^{\,A} \,.\, \supset \,.\, \mathfrak{F}^{\,A}$.

74. *Closure under extension and composition of classes.*— With reference to relations $\bar{K}_i'$ on $\mathfrak{P}'$, $\bar{K}_i''$ on $\mathfrak{P}''$, and $\bar{K}_i$, in particular, $\bar{K}_i'^{\,\prime\prime}$, on $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ we have the following propositions:

(1) $\bar{K}_1'^{\,1} \,.\, \mathfrak{M}'^{\,DK_1'*} \,.\, \bar{K}_1''^{\,1} \,.\, \mathfrak{M}''^{\,DK_1''*} \,.$
$\bar{K}_1^{\,1} \,.\, (\bar{K}_1', \bar{K}_1'', \bar{K}_1)^{12} \,.\, \supset \,.\, (\mathfrak{M}'\mathfrak{M}'')^{\,DK_1*}$.

(2) $\bar{K}_2'^{\,1} \,.\, \mathfrak{M}'^{\,DK_2'*} \,.\, \bar{K}_2''^{\,1} \,.\, \mathfrak{M}''^{\,DK_2''*} \,.$
$\bar{K}_2^{\,1} \,.\, (\bar{K}_2', \bar{K}_2'', \bar{K}_2)^{16} \,.\, \supset \,.\, (\mathfrak{M}'\mathfrak{M}'')^{\,DK_2*}$.

(12) $\bar{K}_{12}'^{\,1} \,.\, \mathfrak{M}'^{\,DK_{12}'*} \,.\, \bar{K}_{12}''^{\,1} \,.\, \mathfrak{M}''^{\,DK_{12}''*} \,.$
$\bar{K}_{12}^{\,1} \,.\, (\bar{K}_{12}', \bar{K}_{12}'', \bar{K}_{12})^{1215} \,.\, \supset \,.\, (\mathfrak{M}'\mathfrak{M}'')^{\,DK_{12}*}$.

(1′) $\bar{K}_1'^{\,1} \,.\, \mathfrak{M}'^{\,DK_1'*} \,.\, \bar{K}_1''^{\,1} \,.\, \mathfrak{M}''^{\,DK_1''*} \,.\, \supset \,.\, \bar{K}_1'^{\,\prime\prime\,1} \,.\, (\mathfrak{M}'\mathfrak{M}'')^{\,DK_1'''*}$.

(2′) $\bar{K}_2'^{\,13} \,.\, \mathfrak{M}'^{\,DK_2'*} \,.\, \bar{K}_2''^{\,13} \,.\, \mathfrak{M}''^{\,DK_2''*} \,.\, \supset \,.\, \bar{K}_2'^{\,\prime\prime\,13} \,.\, (\mathfrak{M}'\mathfrak{M}'')^{\,DK_2'''*}$.

(12′) $\bar{K}_{12}'^{\,12} \,.\, \mathfrak{M}'^{\,DK_{12}'*} \,.\, \bar{K}_{12}''^{\,12} \,.\, \mathfrak{M}''^{\,DK_{12}''*} \,.\, \supset \,.\, \bar{K}_{12}'^{\,\prime\prime\,12} \,.\, (\mathfrak{M}'\mathfrak{M}'')^{\,DK_{12}'''*}$.

Here, in view of preceding propositions, the proposition $i'$ is a corollary of proposition $i$ ($i = 1$; 2; 12).

Proposition $i$ states that its hypothesis implies the conclusion:

($i$) $\mathfrak{P}''' \,.\, \mathfrak{M}'''^{\,LCD} \,.\, \mathfrak{G}_i \equiv [\text{all } \psi^{K_i\mathfrak{M}'\mathfrak{M}''(\mathfrak{M}''')\,.\,B_0\mathfrak{M}'''(p'p'')}] \,.\, \supset \,.$
$(\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''')_* = \mathfrak{G}_i$,

where $\psi$ denotes a function on $\mathfrak{P}'\mathfrak{P}''\mathfrak{P}'''$ and $\mathfrak{M}'''$ denotes a class of functions on $\mathfrak{P}'''$.

Now the class $\mathfrak{G}_i$ contains the class $\bar{\mathfrak{G}}$ :

$$\bar{\mathfrak{G}}_i \equiv [\text{all } \psi^{K_i\mathfrak{M}'\mathfrak{M}''(\mathfrak{M}''') \,.\, B_1\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''' \,.\, B_0\mathfrak{M}'''(p'p'')}],$$

and by preceding propositions we readily see that the class $\bar{\mathfrak{G}}_i$ contains the class $(\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''')_*$.

On the other hand, we are to prove that the class $(\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''')_*$ contains the class $\mathfrak{G}_i$. Now we have

$$(\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''')_* = \bar{\bar{\mathfrak{G}}}_i,$$

where

$$\bar{\bar{\mathfrak{G}}}_i \equiv [\text{all } \psi^{K_i'\mathfrak{M}'(\mathfrak{M}''\mathfrak{M}''') \,.\, K_i''\mathfrak{M}''(\mathfrak{M}'\mathfrak{M}''') \,.\, B_0\mathfrak{M}'''(p'p'')}],$$

for we have

$$(\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''')_* = (\mathfrak{M}'(\mathfrak{M}''\mathfrak{M}''')_*)_* =$$
$$[\text{all } \psi^{K_i'\mathfrak{M}'(\mathfrak{M}''\mathfrak{M}''')_* \,.\, B_0(\mathfrak{M}''\mathfrak{M}''')_*(p')}] =$$
$$[\text{all } \psi^{K_i'\mathfrak{M}'(\mathfrak{M}''\mathfrak{M}''') \,.\, K_i''\mathfrak{M}''(\mathfrak{M}''')(p') \,.\, B_0\mathfrak{M}'''(p'p'')}],$$

in view of the hypotheses :

$$\mathfrak{M}'^{\,DK_i'*}; \quad \mathfrak{M}''^{\,DK_i''*}; \quad \mathfrak{M}'''^{\,LCD},$$

and the remark that $(\mathfrak{M}''\mathfrak{M}''')_*$ is dominated by $\mathfrak{M}''\mathfrak{M}'''$. That $(\mathfrak{M}'\mathfrak{M}''\mathfrak{M}''')_* = \bar{\bar{\mathfrak{G}}}_i$ follows then from the parity of the rôles of $\mathfrak{M}'$, $\mathfrak{M}''$ and the remark :

$$\psi^{K_i''\mathfrak{M}''(\mathfrak{M}'\mathfrak{M}''')} \,.\supset.\, \psi^{K_i''\mathfrak{M}''(\mathfrak{M}''')(p')}.$$

It remains to prove that the class $\bar{\bar{\mathfrak{G}}}_i$ contains the class $\mathfrak{G}_i$, or, that

$$\psi^{K_i\mathfrak{M}'\mathfrak{M}''(\mathfrak{M}''')} \,.\supset.\, \psi^{K_i'\mathfrak{M}'(\mathfrak{M}''\mathfrak{M}''')}.$$

In view of the hypothesis of proposition $i$, this is readily proved for the cases $i = 1$ ; 2. For the case $i = 12$ we have to prove

$$\psi^{K_1\mathfrak{M}'\mathfrak{M}''(\mathfrak{M}''') \,.\, K_2\mathfrak{M}'\mathfrak{M}''(\mathfrak{M}''')} \,.\supset.\, \psi^{K_1'\mathfrak{M}'(\mathfrak{M}''\mathfrak{M}''') \,.\, K_2''\mathfrak{M}''(\mathfrak{M}'\mathfrak{M}''')}.$$

This would be an immediate corollary of the cases $i = 1$ ; 2, except that in the hypothesis of proposition 12 the condition : $(K_2', K_2'', K_2)^6$, of the hypothesis of proposition 2 is replaced by the less exacting condition: $(K'_2, K''_2, K_{12})^5$. This reduction in the hypothesis induces only a slight complication in the proof.

The propositions $i'$ and § $72a12_i$ $(i = 1; 2; 12)$ state that certain genera of systems $(\mathfrak{A}; \mathfrak{P}; K_i; \mathfrak{M})$ are closed under certain operations. There are thus three theorems of which the following is for $i = 12$.

THEOREM I. *The genus of all systems:*

$$(\mathfrak{A}; \mathfrak{P}; K_{12}^{12}; \mathfrak{M}^{D(K_{12})K_{12*}}),$$

*is closed under the combinations*[4] *of the three operations* $A$, $B_1$, $B_2$:

$A$: *$*$-extension of classes* $\mathfrak{M}$;

$B$: *simultaneous composition of classes* $\mathfrak{P}$ *and relations* $K_{12}$ *and*

$B_1$: *multiplication of classes* $\mathfrak{M}$;

$B_2$: *$*$-composition of classes* $\mathfrak{M}$.

*The subgenus of all systems obtained by the combinations of operations* $A$ *and* $B_2$ *is the genus of all systems:*

$$(\mathfrak{A}; \mathfrak{P}; K_{12}^{12}; \mathfrak{M}^{LCD(AK_{12})K_{12*}});$$

*this genus is closed under the operations* $A$, $B_2$ *and their combinations; to this genus belong the systems* I–IV.

Here the properties $(K_{12})$, $(AK_{12})$ are properties implied by the other properties of the respective systems; cf. § $72a11$, 14.

The theorem of § 84 concerning composition of developments $\Delta$ is based on the proposition (12) of § 74.

Similarly from §§ $72a7_\iota$, $72b3''_\iota$ we have the

THEOREM II. *The genus of all systems:*

$$(\mathfrak{A}; \mathfrak{P}; K_\iota; \mathfrak{M}^{DK_\iota}),$$

*is closed under the combinations of the operations* $A$, $B_1$, $B_2$ *(defined above), and* $A_1$, $A_2$:

$A_1$: *extension of classes* $\mathfrak{M}$ *to* $\mathfrak{M}_L$;

$A_2$: *extension of classes* $\mathfrak{M}$ *to* $\mathfrak{M}_*$.

*The subgenus of all systems obtained by the combinations of the operations* $A$, $B_2$ *is the genus of all systems:*

$$(\mathfrak{A}; \mathfrak{P}; K_\iota; \mathfrak{M}^{LCDK_\iota});$$

---

[4] The combinations are understood to include the individual operations. We note that $B_2$ is a combination of $A$ and $B_1$.

*this genus is closed under the operations* $A$, $A_1$, $A_2$, $B_2$ *and their combinations; to this genus belong the systems* I–IV.

These theorems of closure of genera are comparable with theorem IV of § 55 and the theorem of § 84.

### *Systems* ($\mathfrak{A}$; $\mathfrak{P}$; $\Delta$; $\mathfrak{M}$). *Developments* $\Delta$. *Relations* $K_{12}^{\Delta}$. *Classes* $\mathfrak{M}^{\Delta}$. §§ 75–80.

75. *Developments* $\Delta$ *of classes* $\mathfrak{P}$.—A *development* $\Delta$ *of a class* $\mathfrak{P}$ *of elements* is a sequence $\{\Delta^m\}$ of systems $\Delta^m$ of subclasses of $\mathfrak{P}$, each system $\Delta^m$ consisting of a finite number, say $l_m$, of subclasses of $\mathfrak{P}$. The system $\Delta^m$ is *stage* $m$ *of the development* $\Delta$. The sequence $\{l_m\}$ of positive integers is *the sequence of the development* $\Delta$.

There is no question of order but there may be repetitions amongst the $l_m$ classes of stage $m$, and it is to provide for this possibility of repetition that we say *system* $\Delta^m$ instead of *class* $\Delta^m$.

With this understanding, we introduce an *index* $l^m$:

$$l^m = 1;\ 2;\ \cdots,\ l_m,$$

*of stage* $m$, and omit the superscript $m$ of the index $l^m$ when, in conjunction with the index $m$ of the stage, it occurs in the bipartite index $ml$. We denote the $l_m$ subclasses of stage $m$ respectively by the notations:

$$\mathfrak{P}^{ml},\quad [p^{ml}] \qquad (l = 1, 2, \cdots, l_m).$$

Then stage $m$ of the development $\Delta$ is the system:

$$\Delta^m \equiv (\mathfrak{P}^{ml}),$$

and the development $\Delta$ is the system:

$$\Delta \equiv ((\mathfrak{P}^{ml})),$$

of subclasses $\mathfrak{P}^{ml}$ of the class $\mathfrak{P}$.

Admitting the possibility of nullclasses $\mathfrak{P}^{ml}$, we notice as extreme instances the developments $\Delta = \Delta^{\circ}$; $\Delta^{\square}$, where in

$\Delta^{\circ}$: *every class* $\mathfrak{P}^{ml}$ *is the nullclass;*

$\Delta^{\square}$: *every class* $\mathfrak{P}^{ml}$ *is the class* $\mathfrak{P}$.

We recall also the developments:

$$\Delta \equiv \Delta^{\mathrm{I}},\ \Delta^{\mathrm{II}_2},\ \Delta^{\mathrm{III}},\ \Delta^{\mathrm{IV}},$$

of $\mathfrak{P}^{\mathrm{I}}$, $\mathfrak{P}^{\mathrm{II}_2}$, $\mathfrak{P}^{\mathrm{III}}$, $\mathfrak{P}^{\mathrm{IV}}$ respectively, which were defined in § 66 *a*) and § 67 *a*).

A *representative system*:

$$\mathfrak{R}^{\Delta} \equiv ((r^{ml})),$$

of a development $\Delta$ is a system of elements $r^{ml}$ of $\mathfrak{P}$, the element $r^{ml}$ belonging to the class $\mathfrak{P}^{ml}$ of the development $\Delta$. The element $r^{ml}$ is said to *represent* the class $\mathfrak{P}^{ml}$ and its elements $p^{ml}$. It is understood however that only the existent classes $\mathfrak{P}^{ml}$ have representative elements $r^{ml}$.

76. *Various classes derived from a development* $\Delta$. — For a development $\Delta$ *the class* $\mathfrak{P}^{m}$:

$$\mathfrak{P}^{m} \equiv [\,p^{m}\,] \equiv \mathbf{U}\Delta^{m} \equiv \mathbf{U}(\mathfrak{P}^{ml}),$$

is the class of all elements (in notation: $p^{m}$) developed in stage $m$, the least common superclass of the classes of stage $m$, and *the class* $\mathfrak{P}^{\Delta}$:

$$\mathfrak{P}^{\Delta} \equiv [\,p^{\Delta}\,] \equiv \mathbf{U}\Delta \equiv \mathbf{U}((\mathfrak{P}^{ml})) \equiv \mathbf{U}(\mathfrak{P}^{m}),$$

is the class of all elements (in notation: $p^{\Delta}$) developed in the development $\Delta$, the least common superclass of the classes $\mathfrak{P}^{ml}$ of $\Delta$.

*The class* $\mathfrak{P}^{\prime m}$:

$$\mathfrak{P}^{\prime m} \equiv [\,p^{\prime m}\,] \equiv \mathfrak{P} - \mathfrak{P}^{m},$$

is the class of all elements (in notation: $p^{\prime m}$) not developed in stage $m$, and *the class* $\mathfrak{P}_{m}$:

$$\mathfrak{P}_{m} \equiv \mathbf{U}(\mathfrak{P}^{\prime m_0} \mid m_0 \geqq m),$$

is the class of all elements each for some stage $m_0 \geqq m$ not developed, the least common superclass of the classes $\mathfrak{P}^{\prime m_0}$ $(m_0 \geqq m)$.

*As to an element* $p$ *the indices* $l$ *of existent classes of stage* $m$ *are designated by the notations:*

$$g^{r};\ h^{r},$$

ording as the class $\mathfrak{P}^{ml}$ contains or does not contain the ele-
nt $p$. Then *the class* $\mathfrak{P}^{pm}$:

$$\mathfrak{P}^{pm} \equiv \mathbf{U}\,(\mathfrak{P}^{mg^p} \,|\, g^p),$$

the least common superclass of the classes of stage $m$ which
ntain $p$, the class of all elements each of which is by stage $m$
the development *directly connected* with the element $p$, in that
y belong to the same class of stage $m$. The class $\mathfrak{P}_{pm}$:

$$\mathfrak{P}_{pm} \equiv \mathbf{U}\,(\mathfrak{P}^{pm_0} \,|\, m_0 \geqq m),$$

the least common superclass of the classes $\mathfrak{P}^{pm_0} (m_0 \geqq m)$, the
ss of all elements each of which is by some stage $m_0 \geqq m$ of
development directly connected with the element $p$.

77. *The relations* $K^{\Delta}_{1}$, $K^{\Delta}_{2}$, $K^{\Delta}_{12}$ *derived from a development*
*The corresponding properties* $K^{\Delta}_{i}$.—A development $\Delta$
es rise to a *relation* $K_1 \equiv K^{\Delta}_{1}$ and a *relation* $K_2 \equiv K^{\Delta}_{2}$ on
viz., the relations for which

$$K_{pm} \,.\equiv p^{B_0 \mathfrak{P}_m} \,;\; K_{p_1 p_2 m} \,.\equiv .\, p_1^{B_0 \mathfrak{P}_{p_2 m}},$$

more explicitly,

$$K_{pm} \,.\equiv .\, (p,\, m) \ni \exists \;\; m_0 \geqq m \ni p^{-m_0};$$

$$_{p_1 p_2 m} \,.\equiv .\, (p_1, p_2, m) \ni \exists \; (m_0 \geqq m \,.\, l_0 \leqq l_m) \ni (p_1^{m_0 l_0}, p_2^{m_0 l_0}),$$

at is,

$K_{pm}$ *denotes an element* $p$ *for some stage* $m_0 \geqq m$ *not de-*
*oped;*

$K_{p_1 p_2 m}$ *denotes two elements* $p_1$, $p_2$ (*not necessarily distinct*)
*onging to the same class of some stage* $m_0 \geqq m$.

These relations $K^{\Delta}_{1}$, $K^{\Delta}_{2}$ are present simultaneously, and we
ak of the relation $K^{\Delta}_{12}$.

These relations obviously have the properties 1, 2, 4 of § 70.
ey do not necessarily have the properties 3, 5 of § 70; they
y however have either or both of these properties.

In terms of these relations $K^{\Delta}_{1}$, $K^{\Delta}_{2}$ we have, in accordance
h the definitions of § 72, the *bipartite properties:*

$$K^{\Delta}_{1};\; K^{\Delta}_{2},$$

of functions and classes of functions on $\mathfrak{P}$, with their various *derived properties of functions and of classes of functions*, in particular, the properties:

$$K^{\Delta}_{1};\ K^{\Delta}_{2};\ K^{\Delta}_{12};\ K^{\Delta}_{11};\ K^{\Delta}_{21};\ K^{\Delta}_{12*},$$

of classes of functions. (Cf. § 72.)

78. *Developmental systems $\mathfrak{D}(\mathfrak{M})$ relative to a class $\mathfrak{M}$ of functions and a development $\Delta$.* — A *developmental system* $\mathfrak{D}(\mathfrak{M})$:

(1) $$\mathfrak{D}(\mathfrak{M}) \equiv ((\delta^{ml})),$$

is a system of functions $\delta^{ml}$ on $\mathfrak{P}$ for which there exists a representative system $\mathfrak{R}^{\Delta} \equiv ((r^{ml}))$ of the development $\Delta$ such that

(1a) $$e : \supset : \exists\ m_e \ni (m \geqq m_e \,.\, p) \,.\, \supset \,. \quad A(\sum_q \delta_p^{mq} - 1) \leqq e \,.\, A(\sum_q A\delta_p^{mq} - 1) \leqq e;$$

(1b) $$\mu \mathrel{.\,\cdot\,.} \supset \mathrel{.\,\cdot\,.} \exists\ \mu_0 \ni e : \supset : \exists\ m_{\mu e} \ni (m \geqq m_{\mu e} \,.\, p) \,.\, \supset \,. \quad \sum_h A\mu_{pmh}\delta_p^{mh} \leqq eA\mu_{0p}.$$

Here the indices $q$, $h$ are understood to be $q^{p}$, $h^{p}$; and the conditions 1*a*, 1*b* are understood to apply only in so far as such indices are available, that is, for the various stages $m$ the condition 1*a* applies only for those elements $p$ which belong to at least one class of stage $m$, and the condition 1*b* applies only for those elements $p$ which belong not to every existent class of stage $m$. Thus, for the development $\Delta^{11}$ there is only the condition 1*a*, independent of $\mathfrak{M}$, on the developmental system $\mathfrak{D}(\mathfrak{M})$; so that, e. g., the system: $\delta^{m1} = 1$; $\delta^{ml} = 0$ $(l \neq 1)$, is a system $\mathfrak{D}(\mathfrak{M})$ for every $\mathfrak{M}$ for this development. For the development $\Delta^{\circ}$ every system $((\delta^{ml}))$ is for every class $\mathfrak{M}$ a developmental system $\mathfrak{D}(\mathfrak{M})$.

Thus, the system $\mathfrak{D}(\mathfrak{M})$ need not actually depend on the class $\mathfrak{M}$. Indeed, a system $\mathfrak{D}$:

(1′) $$\mathfrak{D} \equiv ((\delta^{ml})),$$

satisfying (1*a*) and

(1′*b*) $$m \,.\, p \,.\, h^{p} \,.\, \supset \,.\, \delta_p^{mh^{p}} = 0,$$

is a system $\mathfrak{D}(\mathfrak{M})$ as to every class $\mathfrak{M}$. Such are the developmental systems $\mathfrak{D}$ of § 66 *c*) and § 67 *a*) for the cases I–IV, the condition (1*a*) being satisfied in virtue of the relation:

$$(1'a) \qquad m \,.\, p^m \,.\, \supset \,.\, \sum_g \delta_p^{\,mg} = \sum_g A\delta_p^{\,mg} = 1.$$

We notice that the systems $\mathfrak{D}(\mathfrak{M})$ are the systems $\mathfrak{D}(A\mathfrak{M})$ and they are the systems $\mathfrak{D}(\mathfrak{A}\mathfrak{M})$. Also, if $\mathfrak{M}_0$ belongs to $\mathfrak{M}$ and $\mathfrak{M}$ is dominated by $\mathfrak{A}\mathfrak{M}_0$, the systems $\mathfrak{D}(\mathfrak{M})$ are the systems $\mathfrak{D}(\mathfrak{M}_0)$. Accordingly, $\mathfrak{M}^{D_1}$ implies that the systems $\mathfrak{D}(\mathfrak{M})$ are the systems $\mathfrak{D}(\mathfrak{M}_L)$, and $\mathfrak{M}^{D}$ implies that the systems $\mathfrak{D}(\mathfrak{M})$ are the systems $\mathfrak{D}(\mathfrak{M}_*)$.

79. *The property* $\Delta$ *of classes* $\mathfrak{M}$ *relative to a development* $\Delta$. — By the mediation of the notion of developmental systems $\mathfrak{D}(\mathfrak{M})$ a development $\Delta$ of the class $\mathfrak{P}$ gives rise to an associated *developmental property* [*] $\Delta$ of classes $\mathfrak{M}$ of functions on $\mathfrak{P}$:

$$\mathfrak{M}^{\Delta} \,.\, \equiv \,.\, \mathfrak{M} \ni \exists\ \mathfrak{D}(\mathfrak{M}) \ni \mathfrak{D}(\mathfrak{M})^{B_0\mathfrak{M}},$$

viz., *a class* $\mathfrak{M}$ *having the property* $\Delta$ *is a class* $\mathfrak{M}$ *such that there exists a developmental system* $\mathfrak{D}(\mathfrak{M})$ (*relative to* $\mathfrak{M}$ *and* $\Delta$) *which belongs to* $\mathfrak{M}$.

We notice the propositions:

$$(1) \qquad \mathfrak{M}^{\Delta} \,.\, \supset \,.\, (\mathfrak{A}\mathfrak{M})^{\Delta};$$

$$(2) \qquad \mathfrak{M}^{D_1\Delta} \,.\, \supset \,.\, \mathfrak{M}_L^{\,D_1\Delta};$$

$$(3) \qquad \mathfrak{M}^{D\Delta} \,.\, \supset \,.\, \mathfrak{M}_*^{\,D\Delta}.$$

80. THEOREM I. *If* $\mathfrak{M}$ *is a class of functions on* $\mathfrak{P}$ *having the dominance property* $D_1$ *and the developmental property* $\Delta$ *associated with a development* $\Delta$ *of* $\mathfrak{P}$, *then a function* $\phi$ *on* $\mathfrak{P}$ *having the property* $K_{12}^{\Delta}\mathfrak{M}$ *is dominated by the class* $\mathfrak{A}\mathfrak{M}$ *and belongs to the class* $\mathfrak{M}_*$, *viz.*,

$$\mathfrak{P} \,.\, \Delta \,.\, \mathfrak{M}^{D_1\Delta} : \supset : \phi^{K_{12}^{\Delta}\mathfrak{M}} \,.\, \supset \,.\, \phi^{B_1\mathfrak{A}\mathfrak{M} \,.\, B_0\mathfrak{M}_*}.$$

Generalizing theorem I in the sense of relativity, we have

THEOREM II. $\mathfrak{P}' \,.\, \Delta' \,.\, \mathfrak{M}'^{\,D_1\Delta'} \,.\, \mathfrak{P}'' \,.\, \mathfrak{M}''^{\,D_1} : \supset :$

$$\phi^{K'{}_{12}^{\Delta'}\mathfrak{M}'(\mathfrak{M}'') \,.\, B_0\mathfrak{A}\mathfrak{M}''(p')} \,.\, \supset \,.\, \phi^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}'' \,.\, B_0(\mathfrak{M}'\mathfrak{M}'')_*}.$$

---

[*] Cf. §§ 66 *d*), 67 *a*).

proceed to the proof of the first theorem.

The development $\Delta$ may be such that for every $m$ there ages $m_0 \geqq m$ consisting entirely of null-classes. In this or every $p$ and $m$ the relation $\overline{K}_{pm}$ holds, and accordingly et that $\phi$ has the property $\overline{K}_1^{\Delta}\mathfrak{M}$ (cf. §§ 77, 72) implies $= 0$, and hence the desired conclusions.

We suppose then that for a certain $m_0$ every stage $m \geqq m_0$ ome existent classes, that is, for some elements $p$ there are s $\mathfrak{P}^{mg}$. (Cf. § 76.)

e class $\mathfrak{M}$ has the developmental property $\Delta$ associated the development $\Delta$. There is a developmental system $) = ((\delta^{ml}))$ which belongs to $\mathfrak{M}$ and there is a representa-ystem $\mathfrak{R}^{\Delta} = ((r^{ml}))$ associated with $\mathfrak{D}(\mathfrak{M})$ (cf. §§ 79, 78, 75). unction $\phi$ has the property $\overline{K}_{12}^{\Delta}\mathfrak{M}$, that is, the properties , $\overline{K}_2^{\Delta}\mathfrak{M}$ (cf. §§ 77, 72). Hence for certain functions $\mu_1$, d stages $m_1$, $m_2$, $m_3$ we have

$$\overline{K}_{pm_1}^{\Delta} . \supset . A\phi_p \leqq A\mu_{1p};$$

$$\overline{K}_{p_1 p_2 m}^{\Delta} . \supset . A(\phi_{p_1} - \phi_{p_2}) \leqq A\mu_{2p_1};$$

$$m \geqq m_3 . p^m . \supset . \sum_{gp} A\delta_p^{mgp} \geqq \tfrac{1}{2}.$$

nsider a definite stage $m$ ($m \geqq m_1, m_2, m_3$) of the develop-$\Delta$ and an element $p$ of $\mathfrak{P}$. If the element $p$ is in no class ge $m$, by (1) we have

$$A\phi_p \leqq A\mu_{1p}.$$

n the other hand, the element $p$ is in a class $\mathfrak{P}^{mg}$ of stage y (2) we have

$$A\phi_p \leqq A\phi_{rml} + A\mu_{2p} \leqq Aa_0 + A\mu_{2p},$$

e $a_0$ is a constant, independent of $p$, such that $A\phi_{rml} \leqq Aa_0$ he various indices $l$ of existent classes of stage $m$.

has the property $D_1$ and the functions $\delta^{ml}$ of stage $m$ ng to $\mathfrak{M}$. Hence for a certain function $\mu$ and constant $\bar{a}$ ave

$$A\mu_1 \leqq A\bar{a}\mu; \qquad A\mu_2 \leqq A\bar{a}\mu;$$

$$A\delta^{ml} \leqq A\bar{a}\mu \qquad (l = 1, \cdots, l_m).$$

Then, for a constant $a$, such that $(2l_m A a_0 + 1) A\bar{a} \leqq Aa$, we have from (3–7) for every element $p$

$$A\phi_p \leqq Aa\mu_p. \tag{8}$$

Accordingly, $\phi$ *is dominated by the class* $\mathfrak{A}\mathfrak{M}$.

Now, introducing for the various stages $m$ $(m \geqq m_0)$ the functions $\theta_m$:

(9)* $$\theta_m \equiv {\sum_l}' \phi_{r^{ml}} \delta^{ml},$$

which evidently belong to $\mathfrak{M}_L$, we proceed to prove that

$$\phi = \underset{m}{\mathrm{L}}\, \theta_m \cdot \quad (\mathfrak{P};\ \mathfrak{M}), \tag{10}$$

and accordingly that $\phi$ *belongs to* $\mathfrak{M}_*$. We are to determine a function $\mu$ and for every $e$ an integer $m_e$ such that

$$m \geqq m_e \,.\, p \,.\supset .\ A(\theta_{mp} - \phi_p) \leqq eA\mu_p. \tag{11}$$

The function $\phi$ is dominated by $\mathfrak{A}\mathfrak{M}$ and has the properties $K_1^{\Delta}\mathfrak{M}$, $K_2^{\Delta}\mathfrak{M}$. Hence, for certain functions $\mu_0, \mu_1, \mu_2$; positive constant $a$; and, for every $e$, integers $m_{1e}$, $m_{2e}$ we have:

$$p \,.\supset .\ A\phi_p \leqq aA\mu_{0p}; \tag{12}$$

$$K^{\Delta}_{pm_{1e}} .\supset .\ A\phi_p \leqq eA\mu_{1p}; \tag{13}$$

$$K^{\Delta}_{p_1 p_2 m_{2e}} .\supset .\ A(\phi_{p_1} - \phi_{p_2}) \leqq eA\mu_{2p_1}. \tag{14}$$

From the properties 1$a$, $b$ of the developmental system $\mathfrak{D}(\mathfrak{M}) \equiv ((\delta^{ml}))$ (cf. § 78) we have for a certain function $\mu_3$ and, for every $e$, certain integers $m_{3e}$, $m_{4e}$ the relations:

$$p \,.\, m \geqq m_{3e} .\supset .\ \sum_h A\mu_{0r^{mh}} \delta_p^{mh} \leqq eA\mu_{3p}; \tag{15}$$

$$p \,.\, m \geqq m_{4e} .\supset .\ A\Big(\sum_g \delta_p^{mg} - 1\Big) \leqq e \,.\ \sum_g A\delta_p^{mg} \leqq 2. \tag{16}$$

Now, if $p$ is an element of no class of a certain stage $m$ $(m \geqq m_0)$, we have from (9)

$$A(\theta_{mp} - \phi_p) \leqq A\phi_p + \sum_h A\phi_{r^{mh}} \delta_p^{mh},$$

---

*In (9) the mark ′ on the sign of summation is to restrict the index $l$ to the indices of existent classes of stage $m$.

so that by (13, 12, 15)

$$(17)\quad \begin{aligned} &p^{\bar{m}} \,.\, (m \geqq m_0,\ m_{1e},\ m_{3e}) \,.\, \supset \,. \\ &\quad A(\theta_{mp}-\phi_p) \leqq eA\mu_{1p} + a\sum_h A\mu_{0r^{mh}}\delta_p^{\,mh} \leqq e(A\mu_{1p} + aA\mu_{3p}). \end{aligned}$$

On the other hand, if $p$ is an element of stage $m$ $(m \geqq m_0)$, it belongs to one or more classes $\mathfrak{P}^{mg}$ of the $l_m$ classes of stage $m$. Consider the representative elements $r^{mg}$ of these classes, denoting by

$$(18)\qquad r_1,\ r_2$$

two of these representative elements such that

$$(19)\qquad \phi_{r_1} \leqq \phi_{r^{mg}} \leqq \phi_{r_2} \quad (g).$$

The pair of representative elements $r_1$, $r_2$ depends on $p$, $m$, $\phi$. There is always at least one such pair whose elements however are not necessarily distinct.

Then, for this element $p$, separating the indices $l$ of stage $m$ appearing in (9) into the indices $g$, $h$, we have:

$$\begin{aligned} \theta_{mp} - \phi_p = {} & \sum_h{}^{*}\, \phi_{r^{mh}}\delta_p^{\,mh} + \sum_g (\phi_{r^{mg}} - \phi_{r_1})\delta_p^{\,mg} \\ & + (\phi_{r_1} - \phi_p)\sum_g \delta_p^{\,mg} + \phi_p\,(\sum_g \delta_p^{\,mg} - 1), \end{aligned}$$

so that by (19)

$$(20)\quad \begin{aligned} &A(\theta_{mp} - \phi_p) \leqq \sum_h A\phi_{r^{mh}}\delta_p^{\,mh} + \\ &\quad [l_m A(\phi_{r_2} - \phi_{r_1}) + A(\phi_{r_1} - \phi_p)]\sum_g A\delta_p^{\,mg} + A\phi_p A(\sum_g \delta_p^{\,mg} - 1). \end{aligned}$$

Hence, since

$$l_m A(\phi_{r_2} - \phi_{r_1}) + A(\phi_{r_1} - \phi_p) \leqq (l_m + 1)A(\phi_p - \phi_{r_1}) + l_m A(\phi_p - \phi_{r_2}),$$

we have from (12, 14, 16, 15) the relation:

$$(21)\quad \begin{aligned} &p^{m} \,.\, (m \geqq m_0,\ m_{2e},\ m_{41},\ m_{4e},\ m_{3e}) \,.\, \supset \,. \\ &A(\theta_{mp} - \phi_p) \leqq a\sum_h A\mu_{0r^{mh}}\delta_p^{\,mh} + 2(2l_m + 1)eA\mu_{2p} + aeA\mu_{0p} \\ &\qquad \leqq e[aA\mu_{3p} + 2(2l_m + 1)A\mu_{2p} + aA\mu_{0p}]. \end{aligned}$$

---

* If there are no indices $h$ (i. e., if the element $p$ belongs to every class of stage $m$) this and the later summations as to $h$ are understood to have the value 0.

In (17, 21) the constant $a$ and the functions $\mu_0, \mu_1, \mu_2, \mu_3$ are four functions of $\mathfrak{M}$ which are independent of $e, p, m$. $\mathfrak{M}$ has the dominance property $D_1$; thus there exists a function $\mu$ of $\mathfrak{M}$ such that $\mathfrak{A}\mu$ dominates $\mu_0, \mu_1, \mu_2, \mu_3$. Accordingly from (17, 21) which hold for every $e$ we readily exhibit for this function $\mu$ and every $e$ an integer $m_e$ such that the required relation (11) holds.

Hence, *every function* $\phi$ *having the property* $K^{\Delta}_{12}\mathfrak{M}$ *is dominated by the class* $\mathfrak{A}\mathfrak{M}$ *and belongs to the class* $\mathfrak{M}_*$, in case the class $\mathfrak{M}$ has the dominance property $D_1$ and the developmental property $\Delta$ associated with a development $\Delta$ of the class $\mathfrak{P}$, with which is associated also the relation $K^{\Delta}_{12}$ underlying the property $K^{\Delta}_{12}\mathfrak{M}$ of the function $\phi$.

Theorem I has been proved. The proof of the more general theorem II follows precisely the lines of the proof of theorem I. We note merely that the representative elements $r_1', r_2'$ corresponding to the representative elements $r_1, r_2$ of (18), depend not only on $p'$ but on $p''$, a fact however which does not interfere with the argumentation since they do not appear in the relation corresponding to (21).

### *Classes* $\mathfrak{F}$ *contained in* $(\mathfrak{M}'\mathfrak{M}'')_*$. *Classes* $\mathfrak{M}$ *with the property* $K^{\Delta}_{12*}$. §§ 81–82.

81. *Classes* $\mathfrak{F}$ *contained in* $(\mathfrak{M}'\mathfrak{M}'')_*$. — As a corollary of the second theorem of § 80 we have the

THEOREM. *The hypothesis:*

$$\mathfrak{P}' \,.\, \Delta' \,.\, \mathfrak{M}'^{D_1\Delta'} \,.\, \mathfrak{P}'' \,.\, \mathfrak{M}''^{D_1} \,.\, \mathfrak{F} \equiv \left[\text{all } \phi^{K'^{\Delta'}_{12}\mathfrak{M}'(\mathfrak{M}'')\,.\,B_0\mathfrak{A}\mathfrak{M}''(p')}\right],$$

*implies the conclusion:*

$$\mathfrak{F}^{B_1\mathfrak{A}\mathfrak{M}'\mathfrak{M}''.B_0(\mathfrak{M}'\mathfrak{M}'')_*}.$$

This theorem is essentially a generalization of the theorems of §§ 65.8, 66.27; cf. § 67 *c*).

82. *Classes* $\mathfrak{M}$ *with the property* $K^{\Delta}_{12*}$. — From the theorem of § 81 and the theorems $(2_{12}, 3_{12})$ of § 73 we have

THEOREM I. *From*

(1) $$\mathfrak{P}' \,.\, \Delta' \,.\, \mathfrak{M}'^{DK'^{\Delta'}_{12}\Delta'}$$

*es that*

$$\mathfrak{P}'' \,.\, \mathfrak{M}''^{LCD} \,.\, \tilde{\mathfrak{F}} \equiv [\text{all } \phi^{K_{12}^{\prime\Delta'}\mathfrak{m}'(\mathfrak{m}'')\,.\,B_0\mathfrak{M}''(p')}]$$

$$\tilde{\mathfrak{F}} = (\mathfrak{M}'\mathfrak{M}'')_+ \,.\, \tilde{\mathfrak{F}}^{LCD\,.\,B_1\mathfrak{M}'\mathfrak{M}''};$$

$$\mathfrak{M}'^{D_0} \,.\, \mathfrak{M}''^{D_0} \,.\supset.\, (\mathfrak{M}'\mathfrak{M}'')_*^{D_0};$$

$$\mathfrak{M}''^{A} \,.\supset.\, (\mathfrak{M}'\mathfrak{M}'')_*^{A};$$

$$\phi^{K_{12}^{\prime\Delta'}\mathfrak{m}'(\mathfrak{m}'')\,.\,B_0\mathfrak{M}''(p')} \,.\supset.\, \phi^{B_1\mathfrak{M}'\mathfrak{M}''},$$

*further*

$$\mathfrak{M}'^{LC},$$

*urs that*

$$\tilde{\mathfrak{F}}^{B_0\mathfrak{M}'(p'')};$$

$$\phi^{K_{12}^{\prime\Delta'}\mathfrak{m}'(\mathfrak{m}'')\,.\,B_0\mathfrak{M}''(p')} \,.\supset.\, \phi^{B_1\mathfrak{M}'\mathfrak{M}''\,.\,B_0\mathfrak{M}'(p'')}.$$

1, 2) implies (6) has been proved in § 80. Denoting for $\equiv K_{12}^{\prime\Delta'}$ the classes $\tilde{\mathfrak{F}}$ of § 73.$2_{12}$, $3_{12}$ by $\tilde{\mathfrak{F}}_2$, $\tilde{\mathfrak{F}}_3$ respectively, the hypotheses 1, (1′), 2 on $\mathfrak{P}'$, $\Delta'$, $\mathfrak{M}'$, $\mathfrak{P}''$, $\mathfrak{M}''$ we obtain , (3′, 6′) by observing that by § 81 $(\mathfrak{M}'\mathfrak{M}'')_*$ contains $\mathfrak{F}$ by (6) is $\tilde{\mathfrak{F}}_2$ which (contains $\tilde{\mathfrak{F}}_3$ which) contains $(\mathfrak{M}'\mathfrak{M}'')_*$. corollaries of theorem I we have by § 72*a*10, 11 the ing theorems.

EOREM II. *From*

$$\mathfrak{P} \,.\, \Delta \,.\, \mathfrak{M}^{DK^{\Delta}_{12}\Delta}$$

*ours that*

$$\mathfrak{M}^{K^{\Delta}_{12*}}.$$

$$\mathfrak{M}_+ \equiv [\text{all } \phi^{K^{\Delta}\mathfrak{m}}] \,.\, (B_0\mathfrak{M}_* \sim K^{\Delta}_{12}\mathfrak{M}) \,.\, \mathfrak{M}_*^{LCDAK^{\Delta}_{12}K^{\Delta}_{12*}\,.\,B_1\mathfrak{M}\mathfrak{M}}.$$

HEOREM III. *From*

$$\mathfrak{P} \,.\, \Delta \,.\, \mathfrak{M}^{LCDK^{\Delta}_{12}\Delta}$$

*ours that*

$$\mathfrak{M}^{AK^{\Delta}_{12*}} \,.\, \mathfrak{M} \equiv [\text{all } \phi^{K^{\Delta}_{12}\mathfrak{m}}] \,.\, (B_0\mathfrak{M} \sim K^{\Delta}_{12}\mathfrak{M}).$$

ese theorems are essentially generalizations of the theorems §§ 65, 66 ; cf. § 67 *c*).

### *Composition of developments* $\Delta$. *Composition theory of classes with the properties* $DK^{\Delta}_{12*}$, $DK^{\Delta}_{12}\Delta$. §§ 83–84.

83. *Composition of developments* $\Delta$. —Relative to a development $\Delta$ of a class $\mathfrak{P}$ of elements we have defined the relations $K^{\Delta}_{i}$ $(i = 1, 2, 12)$ on $\mathfrak{P}$; representative systems $\mathfrak{R}^{\Delta}$; developmental systems $\mathfrak{D}(\mathfrak{M})$; the developmental property $\Delta$ of classes $\mathfrak{M}$. We postpone the consideration of interrelations of developments of a class $\mathfrak{P}$ of elements.

Consider two developments:

$$\Delta' \equiv ((\mathfrak{P}'^{ml'}));\ \Delta'' \equiv ((\mathfrak{P}''^{ml''})),$$

with corresponding sequences:

$$\{l'_m\};\ \{l''_m\},$$

of the respective classes:

$$\mathfrak{P}';\ \mathfrak{P}''.$$

By the composition* for every stage $m$ of the $l'_m$ classes of stage $m$ of $\Delta'$ with the $l''_m$ classes of stage $m$ of $\Delta''$ we secure the

$$l_m \equiv l'_m l''_m$$

classes of stage $m$ of a development $\Delta$, in notation:

$$\Delta'\Delta'' \text{ or } \Delta''',$$

of the class:

$$\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}'',$$

the *composite development* $\Delta'\Delta'' \equiv \Delta'''$ of the developments $\Delta', \Delta''$. For notation it is convenient to use as index $l^m$ or $l$ of stage $m$ the double-index $l'^m l''^m$ or $l'l''$. Then

$$\Delta'\Delta'' \equiv \Delta''' \equiv ((\mathfrak{P}^{ml})) \equiv ((\mathfrak{P}^{ml'l''})),$$

where

$$l_m \equiv l'_m l''_m \quad (m); \qquad \mathfrak{P}^{ml'l''} \equiv \mathfrak{P}'^{ml'}\mathfrak{P}''^{ml''} \quad (ml'l'').*$$

Similarly, from two representative systems:

$$\mathfrak{R}' \equiv ((r'^{ml'})); \qquad \mathfrak{R}'' \equiv ((r''^{ml''})),$$

---

*In the sense of § 51. It is understood that the composite of two classes is non-existent if either component is non-existent.

of $\Delta'$, $\Delta''$ respectively we obtain a *composite system:*

$$\mathfrak{R}'\mathfrak{R}'' \equiv \mathfrak{R}''' \equiv ((r^{ml})) \equiv ((r^{ml'l''})),$$

where

$$r^{ml'l''} \equiv r'^{\,ml'} r''^{\,ml''} \quad (ml'l''),$$

in fact, a representative system of the composite development $\Delta'\Delta'' \equiv \Delta'''$, the *composite representative system* $\mathfrak{R}'\mathfrak{R}'' \equiv \mathfrak{R}'''$ of the representative systems $\mathfrak{R}'$, $\mathfrak{R}''$.

Similarly, from two developmental systems:

$$\mathfrak{D}' \equiv \mathfrak{D}'(\mathfrak{M}') \equiv ((\delta'^{\,ml'})); \qquad \mathfrak{D}'' \equiv \mathfrak{D}''(\mathfrak{M}'') \equiv ((\delta''^{\,ml''})),$$

relative to ($\Delta'$, $\mathfrak{M}'$), ($\Delta''$, $\mathfrak{M}''$) respectively we obtain by multiplication a *composite system:*

$$\mathfrak{D}'\mathfrak{D}'' \equiv \mathfrak{D}''' \equiv ((\delta^{ml})) \equiv ((\delta^{ml'l''})),$$

where

$$\delta^{ml'l''} \equiv \delta'^{\,ml'} \delta''^{\,ml''} \quad (ml'l''),$$

in fact, a developmental system:

$$\mathfrak{D}'\mathfrak{D}'' \equiv \mathfrak{D}''' \equiv \mathfrak{D}'''(\mathfrak{M}'\mathfrak{M}''),$$

as to ($\Delta'\Delta''$, $\mathfrak{M}'\mathfrak{M}''$), the *composite developmental system* $\mathfrak{D}'\mathfrak{D}'' \equiv \mathfrak{D}'''$ of the developmental systems $\mathfrak{D}'$, $\mathfrak{D}''$.

Hence we readily obtain the propositions:

$$\mathfrak{M}'^{\,\Delta'} \,.\, \mathfrak{M}''^{\,\Delta''} \,.\supset.\, (\mathfrak{M}'\mathfrak{M}'')^{\Delta'''};$$

$$\mathfrak{M}'^{\,D\Delta'} \,.\, \mathfrak{M}''^{\,D\Delta''} \,. \quad .\, (\mathfrak{M}'\mathfrak{M}'')^{D\Delta'''} \,.\, (\mathfrak{M}'\mathfrak{M}'')_*^{\,D\Delta'''},$$

where the properties: $\Delta'$; $\Delta''$; $\Delta'''$ of classes of functions on classes: $\mathfrak{P}'$; $\mathfrak{P}''$; $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$, of elements are the developmental properties associated with developments: $\Delta'$; $\Delta''$; $\Delta''' \equiv \Delta'\Delta''$, of the respective classes of elements.

An element $p$ of $\mathfrak{P} \equiv \mathfrak{P}'\mathfrak{P}''$ has the notation $p'p''$. Bearing in mind the notations of §§ 76, 77 and denoting by $K'_i$; $K''_i$; $K_i$ the relations on $\mathfrak{P}'$; $\mathfrak{P}''$; $\mathfrak{P}$ respectively determined* by $\Delta'$; $\Delta''$;

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* It is to be noticed that, as to composition of relations $K_i$ (cf. § 71),

$$K'^{\,\Delta'}_i K''^{\,\Delta''}_i \qquad K^{\Delta'''}_i$$

for $i$ 1, but not necessarily for $i$ 2, 12, since the relation 3 of § 71 does not necessarily hold for ($K'^{\,\Delta'}_i$, $K''^{\,\Delta''}_i$, $K^{\Delta'''}_i$)

$\Delta'\Delta'' \equiv \Delta'''$, we see that

$$(p'p'')^{m} \cdot \sim \cdot p'^{m} \cdot p''^{m}; \qquad (p'p'')^{-m} \cdot \sim \cdot p'^{-m} \smile p''^{-m},$$

and that the relations: $K_i'$; $K_i''$; $K_i$ satisfy the conditions: 1; 2; 4; 5 of § 71.

84. *Closure under extension and composition of classes.*—By collating preceding propositions, we obtain the following theorem of closure comparable with the theorems I, II of § 74 and theorem IV of § 55.

THEOREM. *The respective genera of all systems:*

$$(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M}^{P}),$$

*where*

$$P \equiv DK^{\Delta}_{12}; \quad D\Delta; \quad DK^{\Delta}_{12}\Delta(K^{\Delta}_{12*}); \quad D(K^{\Delta}_{12})K^{\Delta}_{12*},$$

*are closed under the three operations* $A$, $C_1$, $C_2$*:*

$A$*: *-extension of classes* $\mathfrak{M}$*;*

$C$*: simultaneous composition of classes* $\mathfrak{P}$ *and developments* $\Delta$ *and*

$C_1$*: multiplication of classes* $\mathfrak{M}$*:*

$C_2$*: *-composition of classes* $\mathfrak{M}$*,*

*and the combinations*[*] *of these operations.*

*The respective subgenera of all systems obtained by operations* $A$, $C_2$ *and their combinations are*[†] *the respective genera of all systems:*

$$(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M}^{P_0}),$$

*where*

$$P_0 \equiv LCDK^{\Delta}_{12}; \quad LCD\Delta; \quad LCD(A)K^{\Delta}_{12}\Delta(K^{\Delta}_{12*}); \quad LCD(AK^{\Delta}_{12})K^{\Delta}_{12*}.$$

*These genera are closed under the operations* $A$, $C_2$ *and their combinations; to these genera belong the systems* I–IV.

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[*] The operation $C_2$ is a combination of the operations $A$, $C_1$. The closure of the genera under $A$, $C_1$ is in view of the propositions:

ad $A$: § $72a7_{12}$; § 79.3; § $72a7_{12}$, § 79.3; § $72a12_{12}$,

ad $C_1$: § $72b3_{12}$; § 83; § $72b3_{12}$, § 83; § 74.12.

[†] In view of § $44a$I.

Here the properties:

$$K^{\Delta}_{12*};\ K^{\Delta}_{12};\ AK^{\Delta}_{12*};\ AK^{\Delta}_{12},$$

appearing in parentheses in the specification of various composite properties $P$, $P_0$ are implied, in view of the propositions: § 82 II; § $72a11_{12}$; § 82 II, III; § $72a14_{12}$, by the other constituent properties of the respective composite properties.

In this introductory memoir we have considered in particular the following eight fundamental properties:

$$L;\ C;\ D;\ A;\ K_1;\ K_2;\ \Delta;\ K_{12*},$$

of classes $\mathfrak{M}$ of functions on ranges $\mathfrak{P}$. The first four properties are of general reference; the last four properties are with reference to a general development $\Delta$ of the range $\mathfrak{P}$.

In §§ 46–48 we developed the complete existential theory of the first four properties, proving their complete independence and securing functional characterizations of $\mathfrak{P}$ singular; singular or dual; finite. We have found (§ 82 II) that a class $\mathfrak{M}$ having the properties $L$, $C$, $D$, $K_1$, $K_2$, $\Delta$ has the properties $A$, $K_{12*}$. In his Chicago dissertation of 1910 Mr. A. D. PITCHER is developing the complete existential theory of the eight properties, securing characterizations of ranges $\mathfrak{P}$, developments $\Delta$, and classes $\mathfrak{M}$.

*List of logical signs with interpretations.*

| | |
|---|---|
| = | logical identity |
| ⧧ | logical diversity |
| ≡ | definitional identity |
| . ⊃ . | (for every . .) it is true that<br>( ) implies ( )<br>if ( ), then ( ) |
| . ⊃ . | ( ) is implied by ( ) |
| . ∼ . | ( ) is equivalent to ( )<br>( ) implies and is implied by ( ) |
| ⊃ ; ⊂ ; ∼ | implies ; is implied by ; is equivalent to (as relations of properties) |
| Ǝ | there exists a (system ; class ; element ; etc.) |
| ∍ | such that ; where |
| . | and |
| . . ; : : ; ∴ ∴ | signs of punctuation in connection with signs of implication, etc.; the principal implication of a sentence has its sign accompanied with the largest number of punctuation dots |
| | or |
| - | not |
| [ ] | a class of (elements ; functions ; etc.) |
| [all ] | the class consisting of all (elements ; functions ; etc., having a specified property or satisfying a specified condition) |
| ∪ [𝔓] | the least common superclass of the classes 𝔓 of the class [𝔓] of classes |
| ∩ [𝔓] | the greatest common subclass of the classes 𝔓 of the class [𝔓] of classes |

These signs, with the exception of ⧧ ; ≡ ; ∼ ; [ ], are taken from and are used approximately in the sense of G. PEANO's *Formulario Mathematico*, editio V, fasciculo I, 1906.

# PROJECTIVE DIFFERENTIAL GEOMETRY.

BY

E. J. WILCZYNSKI.

## First Lecture.

The group concept, which has become so fundamental in the whole domain of mathematics, furnishes the most convenient basis for the classification of the various kinds of geometry which have arisen. The geometry of the ancients and its modern developments are characterized by the fact that the properties which they discuss are unaltered by any motion in space. These are the so-called metric properties, such as the length of a line, the magnitude of an angle, etc. Projective geometry is based upon a larger group of transformations, which does not leave lengths and angles unchanged, but which does convert every plane into a plane. I intend to confine my attention to the consideration of these two kinds of geometry, although there is at least one other, that of the birational transformations, which is of the utmost importance, and to which the following remarks may also be applied.

The classification of geometries according to their fundamental groups is not the only one which is possible. There are certain properties, let us speak for example of plane curves, which depend merely upon the fact that certain conditions of continuity are fulfilled, that derivatives of a certain order exist, etc. Such are, for example, the properties of the tangent, the osculating circle, etc. These properties, which, since the invention of the calculus, have been studied by the methods which it affords, are known as differential or infinitesimal properties. The infinitesimal properties of a curve do not depend upon the course of a curve as a whole, but merely upon its nature in the immediate vicinity of one of its points. On the other hand a curve may be considered as a whole, as for example, when it is required to find the total number of points in which it intersects a straight line. Let us, for the sake of convenience, speak of such questions as belonging to the realm of integral geometry. We obtain in this way, with our two fundamental groups, four kinds of geometry: differential and integral projective, and differential and integral metrical geometry. Of these only two have been systematically developed, viz.: integral projective and differential metrical geometry. In the nature of things differential geometry is more easily accessible than integral geometry. For, the relation between the two is precisely that of the differential to the integral calculus. From the properties of differential geometry those of integral geometry may be obtained by integration, a process which requires the invention of special methods for every particular case. If, therefore, we can speak of an integral projective geometry as existing it is nevertheless only in a very special sense. In fact, the configurations of projective geometry have in almost all cases, been assumed to be algebraic, a restriction which amounts to an *a priori* integration. Leaving the algebraic cases aside, it is clear, therefore, that integral geometry, taken in the general sense, must be preceded by differential geometry. The metric half of this latter subject has been occupying the attention of mathematicians since the days of MONGE and GAUSS, and, in the hands of their successors, has reached a high degree of perfection. But the same cannot be said for projective differential geometry, which is the subject of these lectures. Of course we find, in the papers devoted to metrical differential geometry, also some results which are of a projective character. A *systematic* treatment, however, of questions of projective differential geometry has been given so far only for curves and ruled surfaces. In this first lecture I shall give a brief account of the theory of curves from this point of view, confining myself to generalities. In the remaining three lectures I shall treat, somewhat more in detail, the theory of ruled surfaces, both on account of the greater novelty and the greater interest of this latter theory.

Consider a homogeneous linear differential equation of the $n$th order

$$(1) \qquad y^{(n)} + \binom{n}{1} p_1 y^{(n-1)} + \binom{n}{2} p_2 y^{(n-2)} + \cdots + p_n y = 0,$$

where

$$(2) \qquad \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k\,!}$$

represents the coefficient of $x^k$ in the expansion of $(1+x)^n$, and where

$$y^{(\lambda)} = \frac{d^\lambda y}{dx^\lambda},$$

while $p_1, \cdots, p_n$ are functions of $x$. The transformation

$$(3) \qquad y = \lambda(x)\eta,$$

where $\lambda(x)$ is an arbitrary function of $x$, transforms (1) into another equation of the same form. A function of the coefficients and of their derivatives, which has the same value for (1) as for any equation obtained from it by a transformation of form (3), shall be called a *seminvariant*. If such an invariant function contains not merely the coefficients but also $y, y', \cdots, y^{(n-1)}$, it shall be called a *semi-covariant*. It is not difficult to show that all seminvariants are functions of the following $n-1$, and of their derivatives:

$$(4) \qquad P_k = e^{\int p_1 dx} \sum_{i=0}^{k} \binom{k}{i} p_i \frac{d^{k-i}\left(e^{-\int p_1 dx}\right)}{dx^{k-i}} \qquad (k = 2, 3, \cdots, n),$$

the first two of which are, explicitly,

$$(5) \qquad P_2 = p_2 - p_1^2 - p_1', \qquad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1''.$$

There are $n-1$ independent semi-covariants, besides $y$ itself, viz.:

$$(6) \qquad y_k = y^{(k)} + \binom{k}{1} p_1 y^{(k-1)} + \binom{k}{2} p_2 y^{(k-2)} + \cdots + p_k y \qquad (k = 1, 2, \cdots, n-1).$$

These semi-covariants are characterized by the property that if $y$ be transformed by a transformation of form (3), and if the corresponding functions for the transformed equation be denoted by $\eta_k$, the relation between $\eta_k$ and $y_k$ is given by the equation

$$y_k = \lambda(x)\eta_k$$

which corresponds to (3), so that $y_1/y$, $y_2/y$, $\cdots$, $y_{n-1}/y$ are *absolute* semi-covariants.

An arbitrary transformation of the independent variable

$$\xi = \xi(x), \tag{7}$$

converts (1) into a linear differential equation of the same form. LIE and STÄCKEL have shown that the combination of (3) and (7) constitutes the most general point transformation which does not change the form or order of the equation. Such combinations of seminvariants and semi-covariants which are left invariant by the transformation (7) are known as *invariants* and *covariants* respectively. For instance $\theta_3 = P_3 - \frac{3}{2}P_2'$ is an invariant.

Let $\mathfrak{y}_1, \cdots, \mathfrak{y}_n$ be the elements of a fundamental system of solutions of equation (1). This equation, being integrated, $\mathfrak{y}_1, \cdots, \mathfrak{y}_n$ will be known as functions of $x$. Interpret $\mathfrak{y}_1, \cdots, \mathfrak{y}_n$ as the homogeneous coordinates of a point $P_\mathfrak{y}$ in a space of $n-1$ dimensions. As $x$ changes, $P_\mathfrak{y}$ will describe a curve $C_\mathfrak{y}$, an *integral-curve* of the differential equation. The equation, however, has an infinity of such integral-curves. For, the $n$ functions

$$\bar{\mathfrak{y}}_k = \sum_{l=1}^{n} c_{kl}\mathfrak{y}_l \qquad (k = 1, 2, \cdots, n),$$

where the determinant of the constants $c_{kl}$ is different from zero, will also constitute a fundamental system of (1); and any fundamental system of (1) may thus be expressed in terms of any other. The properties of the curve $C_\mathfrak{y}$, which are expressed by the coefficients of (1), are not, therefore, characteristic of any particular integral-curve; they are common to them all. Since the transition from one integral-curve to another is made by means of the most general projective transformation of the space considered, these properties are, therefore, common to a curve and all of its projective transformations; *they are projective properties.*

But the representation

$$\mathfrak{y}_k = f_k(x) \qquad (k = 1, 2, \cdots, n),$$

for a curve $C_\mathfrak{y}$ of a space of $n-1$ dimensions contains two arbitrary (non-essential) elements. In the first place, since the coordinates are homogeneous, only the ratios $\mathfrak{y}_1 : \mathfrak{y}_2 : \cdots : \mathfrak{y}_n$ have a geometrical significance. These ratios are not changed by the transformation (3), i. e., $y = \lambda(x)\eta$. In the second place, the independent variable may be arbitrarily transformed without changing the curve. In other words, the integral-curve of (1) is not changed by the combination of the transformations (3) and (7); these transformations only alter the form of its analytical representation. If it be required, therefore, to characterize analytically a geometrical property of the curve $C'_{\mathfrak{y}}$, in a manner which shall be independent of any special representation, it becomes necessary to investigate the *invariants* of equation (1). *Any projective property of the curve will be expressed by an invariant equation or system of equations, and any invariant equation or system of equations will be the adequate analytical expression of some projective property of the curve.*

For $n = 3$ and $n = 4$ we obtain in this way a theory of plane and space curves, which contains a great deal of interesting detail which time will not permit us to expound. The geometrical interpretation of the semi-covariants is essential for this theory. Incidentally it may be mentioned that there is obtained an adequate geometrical interpretation for the reduction of equation (1) to the FORSYTH-LAGUERRE canonical form, which is characterized by the conditions that the coefficients of the $n - 1$th and $n - 2$th derivatives are equal to zero, a reduction which may always be accomplished in an infinity of ways by transformations of the form (3) and (7).

HALPHEN's investigations on the differential invariants of plane and space curves are geometrically identical with the theory which has just been outlined. If we confine ourself to the case of plane curves, the analytical relation between the two theories may be described as follows: Consider the linear differential equation of the third order

$$y^{(3)} + 3p_1y'' + 3p_2y' + p_3y = 0,$$

and let $y_1, y_2, y_3$ be the elements of a fundamental system. Introduce non-homogeneous coordinates by putting

$$x = \frac{y_2}{y_1}, \qquad y = \frac{y_3}{y_1},$$

and choose $x$ as independent variable, so that

$$y = f(x)$$

will be the equation of the plane curve. The invariants will become functions of $y, y', y''$, etc., and these are the differential invariants of HALPHEN. It is easy, therefore, to derive HALPHEN's differential invariants from the invariants of the differential equation. HALPHEN however obtains them in a different way. He starts with the equation $y = f(x)$ of the curve, and determines functions of $y$, $dy/dx$, $d^2y/dx^2$, etc., which are left invariant when $x$ and $y$ are subjected to the most general projective transformation

$$\bar{x} = \frac{a_0 + a_1 x + a_2 y}{c_0 + c_1 x + c_2 y}, \qquad \bar{y} = \frac{b_0 + b_1 x + b_2 y}{c_0 + c_1 x + c_2 y}.$$

This unsymmetrical and unhomogeneous formulation of the problem is manifestly a disadvantage. It is easy enough to obtain the unhomogeneous form when the homogeneous is known, but the inverse process is far more difficult. If, at the time of HALPHEN's first researches, he had been aware of the connection between his differential invariants and the invariants of linear differential equations, he would probably have put them into the homogeneous form. As he did not notice this connection until later, when his point of view had become analytical rather than geometrical, it was left to the lecturer to recast these researches into the more adequate and elegant form. It should be mentioned moreover that, for this purpose, the geometrical theory of the semi-covariants is essential and HALPHEN does not seem to pay any attention to them. Moreover, for the theory of space curves, a general projective theory of ruled surfaces is a prerequisite, a theory not then in existence. These remarks will suffice to explain the relations between HALPHEN and myself in the construction of this theory.

## SECOND LECTURE.

The theory of ruled surfaces is based upon the consideration of a system of differential equations of the form

$$(A)\qquad \begin{aligned} y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0, \\ z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned}$$

which shall be spoken of as *the system* $(A)$, for the sake of brevity. In connection with this system of equations, consider the infinite group $G$ of transformations

$$(G)\qquad \begin{gathered} \eta = \alpha(x)y + \beta(x)z, \qquad \zeta = \gamma(x)y + \delta(x)z, \qquad \xi = f(x), \\ \alpha\delta - \beta\gamma \neq 0, \end{gathered}$$

where $\alpha$, $\beta$, $\gamma$, $\delta$ and $f$ are arbitrary functions of $x$. The transformations of this group transform $(A)$ into another system of the same kind. The problem presents itself: to find the invariants and covariants of the system $(A)$ under the transformations of the group $G$.

We divide the problem of finding these invariant functions into two parts, by considering first a sub-group of $G$, namely that one in which only the dependent variables are transformed, while the independent variable $x$ is left unaltered. The corresponding invariant functions shall be called *seminvariants* and *semi-covariants*. These having been determined, it remains to find such combinations of them as remain unchanged when the independent variable too is transformed in an arbitrary fashion.

The seminvariants are obtained as follows. Put

$$(1)\qquad \begin{aligned} u_{11} &= 2p'_{11} - 4q_{11} + p_{11}^2 + p_{12}p_{21}, \\ u_{12} &= 2p'_{12} - 4q_{12} + p_{12}(p_{11} + p_{22}), \\ u_{21} &= 2p'_{21} - 4q_{21} + p_{21}(p_{11} + p_{22}), \\ u_{22} &= 2p'_{22} - 4q_{22} + p_{22}^2 + p_{12}p_{21}\,; \end{aligned}$$

then

$$(2)\qquad I = u_{11} + u_{22}, \qquad J = u_{11}u_{22} - u_{12}u_{21}$$

are seminvariants, i. e., have the same value for system $(A)$ as for any system obtained from it by a transformation of the form

$$\eta = \alpha(x)y + \beta(x)z, \qquad \zeta = \gamma(x)y + \delta(x)z,$$

where $\alpha, \cdots \delta$ are arbitrary functions of $x$. Put further

$$\begin{aligned}
v_{11} &= 2u'_{11} + p_{12}u_{21} - p_{21}u_{12}, \\
v_{12} &= 2u'_{12} + (p_{11} - p_{22})u_{12} - p_{12}(u_{11} - u_{22}), \\
v_{21} &= 2u'_{21} - (p_{11} - p_{22})u_{21} + p_{21}(u_{11} - u_{22}), \\
v_{22} &= 2u'_{22} - p_{12}u_{21} + p_{21}u_{12}\,;
\end{aligned} \tag{3}$$

and form a third set of four quantities $w_{11}, w_{12}, w_{21}, w_{22}$, from the quantities $v_{ik}$ and $p_{ik}$ in the same way as the quantities $v_{ik}$ are formed from $u_{ik}$ and $p_{ik}$. Then

$$K = v_{11}v_{22} - v_{12}v_{21}, \qquad L = w_{11}w_{22} - w_{12}w_{21} \tag{4}$$

are two further seminvariants, and *all seminvariants are functions of $I$, $J$, $K$, $L$ and of the derivatives of these quantities.*

The invariants are functions of the seminvariants. Those which are fundamental are:

$$\begin{aligned}
&\theta_4 = I^2 - 4J, \qquad \theta_{4.1} = 8\theta''_4\theta_4 - 9(\theta'_4)^2 + 8I\theta_4^2, \\
&\theta_{10} = \theta_4(K - I'^2) + \tfrac{1}{4}(\theta'_4)^2, \\
&\theta_9 = \Delta,
\end{aligned} \tag{5}$$

where

$$\Delta = \begin{vmatrix} u_{11} - u_{22} & u_{12} & u_{21} \\ v_{11} - v_{22} & v_{12} & v_{21} \\ w_{11} - w_{22} & w_{12} & w_{21} \end{vmatrix}, \tag{6}$$

the expression of which in terms of $I$, $J$, $K$, $L$ is not rational. *All other invariants may be expressed as functions of these four and of certain others obtained from them by a certain process which involves differentiation.* The index, in the above notation, indicates the weight of the invariant, except in the case of $\theta_{4.1}$ which is of weight ten. If $\bar{\theta}_m$ denotes the result of the general transformation of the group $G$ upon an invariant $\theta_m$ of weight $m$, we have

$$\bar{\theta}_m = \frac{1}{(\xi')^m}\theta_m. \tag{7}$$

The fundamental covariants may be obtained as follows.

Put

(8) $$\rho = 2y' + p_{11}y + p_{12}z, \qquad \sigma = 2z' + p_{21}y + p_{22}z;$$

then

(9) $$C_1 = P = z\rho - y\sigma$$

is a covariant of weight one. Put further

(10) $$\begin{aligned} C &= u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz, \\ E &= v_{12}z^2 - v_{21}y^2 + (v_{11} - v_{22})yz. \end{aligned}$$

Then

(11) $$C_2 = C, \qquad C_7 = \theta_4 E - \theta_4' C$$

are two further covariants. Put finally

(12) $$\begin{aligned} \alpha &= 2(u_{11} - u_{22})\rho + 4u_{12}\sigma + \tfrac{1}{2}(v_{11} - v_{22})y + v_{12}z, \\ \beta &= 4u_{21}\rho - 2(u_{11} - u_{22})\sigma + v_{21}y - \tfrac{1}{2}(v_{11} - v_{22})z; \end{aligned}$$

then

(13) $$C_3 = \alpha z - \beta y$$

is a fourth covariant. All of these covariants are quadratic; their weight is indicated by their index. All other covariants may be expressed in terms of these four and of invariants.

We proceed to consider the solutions of a system of form ($A$). Let the functions $p_{ik}$ and $q_{ik}$ be analytic in the vicinity of $x = x_0$. Then, there will exist two functions $y$ and $z$, analytic in the vicinity of $x = x_0$, which satisfy the differential equations, and which, together with their first derivatives, assume arbitrarily prescribed values for $x = x_0$. Such a system of two functions, involving four arbitrary constants, shall be said to constitute a system of general solutions of system ($A$).

Now let $(y_i, z_i)$ for $(i = 1, 2, 3, 4)$ be any four systems of solutions of ($A$). Then, denoting by $c_1, c_2, c_3, c_4$ four arbitrary constants,

(14) $$y = \sum_{i=1}^{4} c_i y_i, \qquad z = \sum_{i=1}^{4} c_i z_i$$

will also form a simultaneous system of solutions. Moreover, from (14) and

(15) $$y' = \sum_{i=1}^{4} c_i y_i', \qquad z' = \sum_{i=1}^{4} c_i z_i',$$

the constants $c_1 \cdots c_4$ can be determined in such a way as to give arbitrary constant values to $y, z, y', z'$ for $x = x_0$, provided that the determinant

$$(16) \qquad D = \begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ z_1' & z_2' & z_3' & z_4' \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

does not vanish for $x = x_0$. If, therefore, $D$ is not identically zero, we can express a *general* system of solutions in terms of $y_1, \cdots, y_4$ and $z_1, \cdots, z_4$ by means of (14). We shall, therefore, speak of four pairs of solutions $(y_i, z_i)$, for which the determinant $D$ does not vanish, as a *fundamental system* of simultaneous solutions.

We may express the condition $D \neq 0$ in another way. If $D = 0$ it is possible to find four functions $\lambda, \mu, \nu, \rho$ of $x$, so that the four equations

$$(17) \qquad \lambda y_k + \mu y_k' + \nu z_k + \rho z_k' = 0 \qquad (k = 1, 2, 3, )4$$

may be verified. If $(y_k, z_k)$ form a fundamental system of solutions, it must therefore be impossible to find functions $\lambda, \mu, \nu, \rho$ so as to satisfy (17).

It may be shown, without any difficulty, that conversely any four pairs of functions $(y_k, z_k)$ determine uniquely a system of form $(A)$, of which they are a fundamental system of solutions, provided that their determinant $D$ does not vanish identically, i. e., provided that they do not satisfy a system of equations of the form (17).

We are now prepared to see what all this has to do with the theory of ruled surfaces. Let us interpret $(y_1, \cdots, y_4)$ and $(z_1, \cdots, z_4)$ as the homogeneous coordinates of two points $P_y$ and $P_z$ of space. As $x$ changes, $P_y$ and $P_z$ describe two curves $C_y$ and $C_z$; the points of these curves, moreover, are put into a definite correspondence with one another, those being corresponding points which belong to the same value of $x$. Construct the tangents of these two curves at corresponding points.

In general they will not intersect; they will, if, and only if, the determinant $D$ is equal to zero. *In order, then, that the curves $C_y$ and $C_z$ may be the integral curves of a system of form* ($A$), *it is necessary and sufficient that the tangents of the two curves, constructed at corresponding points, shall not intersect.*

Join the points $P_y$ and $P_z$ by a straight line $L_{yz}$. The locus of these lines is a ruled surface $S$, the integrating ruled surface of system ($A$). The transformation

$$\eta = \alpha(x)y + \beta(x)z, \qquad \zeta = \gamma(x)y + \delta(x)z, \qquad \alpha\delta - \beta\gamma \neq 0,$$

which is contained in the group $G$, transforms the points $P_y$ and $P_z$ into two other points $P_\eta$ and $P_\zeta$ of the line $L_{yz}$. Since the functions $\alpha, \cdots, \delta$ are arbitrary, it is possible in this way to convert the curves $C_y$ and $C_z$ into any other two curves upon the integrating ruled surface $S$. The correspondence of the points $P_\eta$ and $P_\zeta$ remains such that the line joining corresponding points is a generator of the ruled surface $S$.

A transformation of the form

$$\xi = f(x),$$

where $f(x)$ is an arbitrary function, changes the parametric representation of the curves in the most general way, without altering either the curves themselves or their point to point correspondence. The transformations of the group $G$, therefore, leave the ruled surface $S$ invariant.

The ruled surface $S$, however, is not unique. For, on account of the fact that ($A$) is a system of linear homogeneous equations, it is evident that any projective transformation of $S$ will be an integrating ruled surface of the system. Let us speak of two systems of form ($A$) as *equivalent* if they can be transformed into each other by a transformation of the group $G$. Then we may make the following statement:

*If two systems of differential equations of form* ($A$) *are equivalent, their integrating ruled surfaces are projective transformations of each other. Moreover, if the fundamental systems of solutions be properly selected, the ruled surfaces coincide. Conversely, if the ruled surfaces of two such systems coincide,* *the systems are equivalent. The integrating ruled surface of a system of form (A) is never a developable.*

It is easy to show that any non-developable ruled surface may be defined by a system of form (*A*). The general theory of such systems of differential equations is, therefore, equivalent to the general theory of ruled surfaces.

*Any equation or system of equations between* $p_{ik}$, $q_{ik}$, $p'_{ik}$, *etc., which remains invariant for all transformations of the group* $G$, *expresses a projective property of the integrating ruled surface.*

For, such equations remain unchanged whatever may be the two curves $C_y$ and $C_z$ upon $S$ which are taken as fundamental curves, and whatever may be the independent variable. They express, therefore, properties of the surface itself, independent of any special method of representation. These properties are projective because the coefficients of (*A*) are left invariant by any projective transformation. Conversely, any projective property of a ruled surface can be expressed by an invariant equation, or system of equations.

It remains to introduce the principle of duality. At corresponding points $P_y$ and $P_z$ of the two fundamental curves $C_y$ and $C_z$, let us construct the planes $p_y$ and $p_z$ which are tangent to the ruled surface $S$. They will intersect along the straight line $L_{yz}$ which joins $P_y$ and $P_z$. The four pairs of coordinates, determining these planes $p_y$ and $p_z$, will form a simultaneous fundamental system of solutions for a new system of differential equations, which shall be called the *adjoint* of (*A*). This system may be written as follows:

$$(18)\quad \begin{aligned} &U''+p_{11}U'+p_{12}V'+[q_{11}+\tfrac{1}{4}(u_{11}-u_{22})]U+(q_{12}+\tfrac{1}{2}u_{12})V=0,\\ &V''+p_{21}U'+p_{22}V'+(q_{21}+\tfrac{1}{2}u_{21})U+[q_{22}+\tfrac{1}{4}(u_{22}-u_{11})]V=0. \end{aligned}$$

The invariants of even weight are identical for (18) and for (*A*), while those of odd weight differ in sign only.

Systems (*A*) and (18) are identical if

$$(19)\qquad u_{12}=u_{21}=u_{11}-u_{22}=0.$$

From the relations between the solutions of the two systems it follows that, in this case, the ruled surface is a quadric. *Th* *conditions* (19) *are, therefore, characteristic of such systems* (*A* *whose integrating ruled surfaces are quadrics. Such system* *coincide with their adjoints.*

A further result of fundamental importance, which is simple consequence of the relations between the solutions c a system of form ($A$) and of its adjoint, is the following.

*If, in a system of form* ($A$), $p_{12} = p_{21} = 0$, *its integral curve* *are asymptotic lines on its integrating ruled surface.*

From this result may be deduced the theorem of Pau Serret: *The double-ratio of the four points, in which a movin* *generator intersects four fixed asymptotic curves of a ruled surfac* *is constant.*

The system ($A$) being given, its invariants may, of cours be computed. But, with a certain restriction, the converse also true. Upon this fact is based the proof of the followin theorem which, from our point of view, is the fundament theorem of the theory of ruled surfaces.

*If* $\theta_4$, $\theta_{4.1}$, $\theta_9$ *and* $\theta_{10}$ *are given as arbitrary functions of* *provided however that* $\theta_4$ *and* $\theta_{10}$ *are not identically equal to zer* *they determine a ruled surface uniquely except for projective tran* *formations.*

We shall not stop to deduce further consequences from th theorem. It may be noted, however, that it corresponds the fundamental theorem of the metrical theory of surface which states that the two fundamental quadratic forms suffic to determine a surface except for its position in space.

## THIRD LECTURE.

Important new ideas are suggested by the problem of provi ing a suitable geometrical interpretation for the covariants an semi-covariants. Let us begin with the covariant

$$P = z\rho - y\sigma, \tag{1}$$

where

$$\rho = 2y' + p_{11}y + p_{12}z, \qquad \sigma = 2z' + p_{21}y + p_{22}z. \tag{2}$$

If we substitute $y = y_k$, $z = z_k$ $(k = 1, 2, 3, 4)$ into these expressions, the two sets of four quantities $\rho_k$ and $\sigma_k$ may again be interpreted as the homogeneous coordinates of two points $P_\rho$ and $P_\sigma$. It is at once apparent that $P_\rho$ and $P_\sigma$ will be points of the planes tangent to the ruled surface $S$ at $P_y$ and $P_z$ respectively. But we may describe the position of these points more completely.

In the first place it may be shown that, if $P_y$ and $P_z$ are converted into two other points $P_{\bar{y}}$ and $P_{\bar{z}}$ of the line $L_{yz}$, by the equations

$$y = \alpha\bar{y} + \beta\bar{z}, \qquad z = \gamma\bar{y} + \delta\bar{z},$$

then $P_\rho$ and $P_\sigma$ are transformed cogrediently into $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$, where

$$\rho = \alpha\bar{\rho} + \beta\bar{\sigma}, \qquad \sigma = \gamma\bar{\rho} + \delta\bar{\sigma},$$

i. e., into two points of the line $L_{\rho\sigma}$ which joins $P_\rho$ to $P_\sigma$. We find, therefore, a line $P_\rho P_\sigma$ which has a definite point-to-point correspondence with the line $P_y P_z$, the generator of $S$.

This point-to-point correspondence is a very simple one. Suppose that the system $(A)$ has been reduced to such a form that $p_{12} = p_{21} = 0$, so that the curves $C_y$ and $C_z$ are asymptotic curves on $S$. Equations (2) show that $P_\rho$ and $P_\sigma$ will then be points upon the tangents of these curves. Let us speak of the tangents to the asymptotic curves of $S$ as its *asymptotic tangents*. The totality of these lines along a given generator $g$ of a ruled surface constitutes a set of generators of a hyperboloid $H$, the *osculating hyperboloid* of $S$ along $g$. This hyperboloid may also be defined as passing through $g$ and two other generators of $S$ infinitesimally close to it. The generators of $H$ which belong to the same set as $g$ or $L_{yz}$ may be called generators of the first kind. The asymptotic tangents of $S$ along $g$ will then constitute the second set of generators of $H$. We may now say that *the line $L_{\rho\sigma}$ is a generator of the first kind upon the hyperboloid $H$ which osculates $S$ along $L_{yz}$. The correspondence between the points of $L_{yz}$ and $L_{\rho\sigma}$ is such that the lines which join corresponding points are the generators of the second kind on the osculating hyperboloid.*

It is the totality of the generators of the first set upon $H$, rather than a particular one, which is of importance in this connection. In fact, if the independent variable be changed by putting

$$\xi = \xi(x),$$

$\rho$ and $\sigma$ are converted into

$$(3) \qquad \bar{\rho} = \frac{1}{\xi'}(\rho + \eta y), \qquad \bar{\sigma} = \frac{1}{\xi'}(\sigma + \eta z),$$

where

$$(3a) \qquad \eta = \frac{\xi''}{\xi'}.$$

The factor $1/\xi'$ is of no importance, since we are concerned only with the ratios $\rho_1 : \rho_2 : \rho_3 : \rho_4$, etc. By choosing $\eta$ conveniently as a function of $x$, we may clearly make $L_{\bar{\rho}\bar{\sigma}}$ coincide with any generator of the first kind on $H$ excepting only $g$ itself.

As $x$ changes, the line $L_{\rho\sigma}$ describes a ruled surface $S'$. The generators of this surface belong to the congruence $\Gamma$, which is formed by the generators of the first kind on the single infinity of hyperboloids which osculate $S$. There is one generator of $S'$ upon each of these hyperboloids. Clearly, by an appropriate choice of the independent variable, the surface $S'$ may be made to coincide with any ruled surface of the congruence $\Gamma$ which has one of its generators on each of the osculating hyperboloids. We, therefore, speak of $S'$ as being the *derivative of $S$ with respect to $x$*. The derivative ruled surface may serve as an image of the independent variable. This image changes, in general, whenever the independent variable is transformed. It does not change, however, when $x$ is made to undergo a linear transformation

$$\xi = ax + b,$$

since such a transformation does not change the value of $\eta$ as defined by (3a). A linear transformation of the independent variable is, therefore, without any geometrical significance.

We leave the covariant $P$ and the congruence $\Gamma$ which it defines, in order to return to it a little later. The covariant

$$(4) \qquad C = u_{12}z^2 - u_{21}y^2 + (u_{11} - u_{22})yz$$

may be decomposed into two factors linear and homogeneous in $y$ and $z$. It defines, therefore, two points on every generator of the ruled surface $S$. If the locus of these points be chosen as fundamental curves, the corresponding system $(A)$ is characterized by the conditions $u_{12} = u_{21} = 0$. Geometrically these two curves are characterized as follows. Four lines in space have two real, imaginary or coincident straight line intersectors. Consider a fixed generator $g$, and three other generators $g_1$, $g_2$, $g_3$, of the ruled surface $S$, together with their two intersectors. Let the generators $g_1$, $g_2$, $g_3$ approach $g$ as a limit. The intersectors of $g$, $g_1$, $g_2$, $g_3$ will, in general, approach limits $f'$ and $f''$, which will determine two points upon $g$. We may say briefly that these are the points of $g$ at which tangents to $S$ may be constructed which have four consecutive points in common with the surface. Following a nomenclature due to CAYLEY, we shall call these points the *flecnodes* of $g$. The four-point-tangents shall be called the *flecnode tangents;* the locus of the flecnodes on $S$ is its *flecnode curve* and the locus of the flecnode tangents, a ruled surface of two sheets, is the *flecnode surface of* $S$. It may be remarked at once that the flecnode tangent is not, in general, tangent to the flecnode curve, never, in fact, unless the latter degenerate into a straight line.

Since the discriminant of the covariant $C$ is equal to $\theta_4$, the significance of the condition $\theta_4 = 0$ becomes apparent. We may recapitulate as follows:

*The flecnode curve is determined by factoring the covariant $C$. Its two intersections with the generators of the ruled surface are distinct if $\theta_4 \neq 0$; they coincide if $\theta_4 = 0$. If the integral curves $C_y$ and $C_z$ of a system of form $(A)$ are the two branches of the flecnode curve, this system of differential equations is characterized by the conditions*

$$u_{12} = u_{21} = 0.$$

It is important for many purposes to establish the relations which exist between a ruled surface and the fundamental configurations of line geometry. This may be accomplished by setting up the linear homogeneous differential equation of the sixth order which is satisfied by the Plückerian coordinates of the generators of the surface. The result may be recapitulated in the following theorem.

*The necessary and sufficient condition for a ruled surface belonging to a single linear complex, which is not special, are*

$$\theta_4 \neq 0, \qquad \Delta = 0, \qquad \theta_{10} \neq 0,$$

*while all of the minors of the second order in* $\Delta$ *do not vanish. If* $\theta_{10} = 0$, *while the other conditions remain the same, the complex is special. The surface belongs to a linear congruence with distinct directrices if all of the minors of the second order in* $\Delta$ *vanish, while* $\theta_4$ *is different from zero. The directrices of the congruence coincide if* $\theta_4$ *also vanishes. In this latter case the surface is, or is not a quadric according as the equations*

$$u_{11} - u_{22} = u_{12} = u_{21} = 0$$

*are, or are not satisfied.*

We return to the consideration of the congruence $\Gamma$. The first question, which suggests itself, concerns its developables and its focal surface. The differential equations of the surface $S'$, the derivative of $S$ with respect to $x$, show that this surface is developable if and only if the seminvariant $J$ is equal to zero. The complete answer to the question in regard to the developables of the congruence based upon this remark is as follows.

*The congruence contains two families of* $\infty^1$ *developables, which coincide if and only if* $\theta_4 = 0$, *i. e., if and only if the two branches of the flecnode curve of* $S$ *coincide. To determine any developable surface of the congruence, it is necessary and sufficient to find a solution of the equation*

$$4\{\xi, x\}^2 + 2I\{\xi, x\} + J = 0,$$

*where* $\{\xi, x\}$ *denotes the Schwarzian derivative of* $\xi$ *with respect to* $x$, *and to take this solution* $\xi = \xi(x)$ *as the independent variable of the defining system of differential equations. The derivative of* $S$ *with respect to* $\xi$ *will then be a developable surface, and all developables of the congruence may be obtained in this way. Moreover, any four developables of the same family intersect all of the asymptotic tangents of* $S$ *in point-rows of the same cross-ratio.*

The locus of the cuspidal edges of these two families of developables is the focal surface of the congruence. It may be easily shown that *the focal surface of the congruence coincides with the flecnode surface of S.* For this reason we shall speak of the congruence Γ as the *flecnode congruence* of S.

Each sheet of the flecnode surface of S, has S itself as one of the sheets of *its* flecnode surface. The second sheet of its flecnode surface, however, is never a surface of the congruence Γ except in degenerate cases. It is possible, nevertheless, to find two families of $\infty^2$ non-developable surfaces of the congruence Γ each of which has one of the branches of its flecnode curve on one of the sheets of the flecnode surface of S. Both branches of the flecnode curve of a surface of Γ can be situated upon the flecnode surface of S only if S belongs to a linear complex.

## FOURTH LECTURE.

If we trace any curve on a ruled surface S, there will correspond to it a perfectly definite curve on the derivative $S'$; the lines joining corresponding points being the asymptotic tangents of S. The investigation of such correspondences gives rise to a number of important results. It may be shown that there exists a single infinity of ruled surfaces in the flecnode congruence two of whose asymptotic lines correspond in this way to the flecnode curve on S. In other words this may be expressed as follows:

*If S is a ruled surface with two distinct branches to its flecnode curve and not belonging to a linear congruence, there exists just a single infinity of ruled surfaces in the congruence Γ, whose intersections with the two sheets of the flecnode surface of S are asymptotic lines upon them. They are the derivatives of S, when the independent variable is so chosen as to make the seminvariant I vanish. Moreover, the point-rows, in which any four of these surfaces intersect the asymptotic tangents of S, all have the same anharmonic ratio.*

A system of form (A) may always be reduced to the so-called *canonical* form, for which

$$p_{ik} = 0, \qquad q_{11} + q_{22} = 0.$$

The significance of this reduction is now apparent. The fundamental curves $C_y$ and $C_z$ are any two asymptotic curves on $S$, and the independent variable $x$ is chosen in such a way that the flecnode surface of $S$ intersects the derivative $S'$ of $S$ with respect to $x$ along a pair of asymptotic curves.

If $\theta_4 = 0$ the family of surfaces just characterized coincides with the (single) family of developables of the congruence.

If the ruled surface $S$ has two straight line directrices, the reduction to the canonical form has a different significance. There exist, in that case, $\infty^2$ ruled surfaces of the congruence $\Gamma$ whose asymptotic lines correspond to those of $S$. These may be arranged in $\infty^1$ families of $\infty^1$ surfaces. One of these families, not very essentially distinguished however from any of the others, is obtained by the reduction to the canonical form.

The process of forming the derivative of a ruled surface with respect to an independent variable may be repeated. We thus obtain $S''$ the second derivative of $S$ with respect to $x$. This may coincide with $S$ itself. We then have the following result: *Every ruled surface which has a second derivative coinciding with $S$ itself may be defined by a system of form (A) for which $p_{ik} = 0$, and for which the quantities $q_{ik}$ are constants. If the independent variable is so chosen that the second derivative $S''$ coincides with $S$, the first derivative $S'$ is a projective transformation of the original surface, and its asymptotic lines correspond to those of $S$.*

Such surfaces are either of the form

$$x_1^{\lambda} x_4^{\mu} - x_2^{\lambda} x_3^{\mu} = 0,$$

where $\lambda$ and $\mu$ are constants, or of the form

$$2q(x_1 x_4 - x_2 x_3) = x_3 x_4 \log \frac{x_3}{x_4},$$

where $q$ is a constant. *All of the asymptotic lines of such a surface are anharmonic curves with the same invariants.*

We have seen that the derivative $S'$, of $S$ with respect to $x$, is cut by the flecnode surface of $S$ along a pair of asymptotic lines if $I = 0$. But this relation turns out to be reciprocal, i. e., the surface $S'$ cuts out an asymptotic curve upon each sheet of the flecnode surface of $S$. The asymptotic lines upon the two sheets of the flecnode surface, therefore, correspond to each other. Since the flecnode surface is also the focal surface of the congruence $\Gamma$, we see that *the flecnode congruence is a so-called W-congruence.*

It will be clear without lengthy explanation what we mean by the osculating linear complex of a ruled surface. Every linear complex gives rise to a point-plane correspondence. Thus, there corresponds, in the osculating linear complex, to every point of the generator $g$ of a ruled surface, a plane containing $g$. There corresponds to every point of $g$ another plane containing $g$, namely the plane which is tangent to $S$ at that point. There will be two points on $g$ (in general) at which these two planes coincide. We shall speak of these as the *complex points* of $g$, and of their locus on $S$ as the *complex curve. The complex curve, like the flecnode curve, intersects every generator in two points. These four points form a harmonic group on every generator.* Of course the complex curve may be determined by factoring a quadratic covariant, viz.:

$$[(u_{11}-u_{22})v_{12}-(v_{11}-v_{22})u_{12}]z^2+[(u_{11}-u_{22})v_{21}-(v_{11}-v_{22})u_{21}]y^2 \\ +2(u_{12}v_{21}-u_{21}v_{12})yz.$$

*The complex curve becomes an asymptotic curve if the surface $S$ belongs to a linear complex.* The covariant $C_7$ of Lecture II determines a pair of points which divides both flecnodes and complex points harmonically. Of course not all three point-pairs can be real simultaneously.

Let $H$ be the hyperboloid which osculates the ruled surface $S$ along one of its generators $g$. Let $g'$, a generator of the first set on $H$, be the corresponding generator of the derived ruled surface $S'$, and let $H'$ be the hyperboloid which osculates $S'$ along $g'$. Since $g'$ is situated entirely upon $H'$ as well as upon $H$, the rest of the intersection of these two hyperboloids is, in general, a space cubic. This cubic is called the *derivative cubic.* We obtain, in this way, associated with every ruled surface, surfaces containing a single infinity of space cubics. The derivative cubic degenerates only if the two branches of the flecnode curve of $S$ coincide, if $S$ has a straight-line directrix, or if the derivative surface $S'$ is one of the developables of the congruence $\Gamma$. The relations of the derivative cubic to the ruled surface are numerous and of considerable interest. We shall mention only the following theorem.

Let the derivative surface $S'$ be one of the $\infty^1$ surfaces of the congruence $\Gamma$ which intersects the flecnode surface along an asymptotic curve. The derivative cubic will then intersect the generator of $S$ in its complex points.

If the two branches of the flecnode curve of $S$ coincide, there is a derivative conic to consider instead of a derivative cubic.

The flecnode curve being of fundamental importance for the projective theory of ruled surfaces, it becomes essential to find out to what extent it may be arbitrarily assigned. We find the following results: *An arbitrary space curve being given, it can be considered as one branch of the flecnode curve of an infinity of ruled surfaces into whose general expression there enters an arbitrary function. Two curves taken at random cannot be connected, point to point, in such a way as to constitute the complete flecnode curve upon the ruled surface thus generated.*

From any ruled surface a single infinity of others can be derived, each of which has one branch of its flecnode curve in common with it. This gives rise to an interesting configuration which we proceed to describe.

Let $C$ be a branch of the flecnode curve of a ruled surface $S$, and let the developable surface formed by its tangents be called its *primary developable*. There exists another important developable surface containing $C$, which we shall call its *secondary developable*, as indicated in the following theorem.

*If at every point of the flecnode curve of $S$ there be drawn the generator of the surface, the flecnode tangent, the tangent of the flecnode curve, and finally the line which is the harmonic conjugate of the latter with respect to the other two, the locus of these last lines is a developable surface, the secondary developable of the flecnode curve.*

*We can find a single infinity of ruled surfaces, each having one branch of its flecnode curve in common with that of S. This family of* $\infty^1$ *surfaces may be described as an involution, of which any surface of the family and its flecnode surface form a pair. The primary and secondary developables of the branch of the flecnode surface considered, are the double surfaces of this involution. In fact, the generators of these surfaces, at every point of their common flecnode curve, form an involution in the usual sense.*

In the course of these few lectures I have been able to touch merely the most important points of this new theory. Many problems remain to be solved. The field is a fruitful one whose cultivation promises further valuable results. I am, at present, occupied with the construction of a general theory of surfaces from the point of view of projective differential geometry. I hope that I may be in a position to discuss that more general subject with you upon a future occasion. For the present I must close, thanking the Society and all of you for the precious opportunity of explaining this theory to an audience of competent mathematicians.

# SELECTED TOPICS IN THE THEORY OF BOUNDARY VALUE PROBLEMS OF DIFFERENTIAL EQUATIONS.

BY

MAX MASON.

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The title of these lectures may, I hope, be sufficient apology for their fragmentary character. The theory of boundary value problems is a field too extended to admit of systematic discussion in four lectures.* But the interesting nature of the problems and their importance in the applications form perhaps a sufficient reason for a presentation of even a fragmentary treatment. I have chosen the topics and the methods of treatment entirely according to my individual interest, feeling that a personal point of view is expected, and perhaps desirable, in the Colloquium Lectures of the Society.† I have attempted however to link the separate topics together as far as possible, and to produce some mild degree of homogeneity, even though this course necessitates the discussion of some very familiar topics.

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* Excellent systematic accounts of the progress in this field up to the year 1900 are given by BÔCHER and SOMMERFELD in the *Encyklopädie der Mathematischen Wissenschaften.*

† I regret that the application of integral equations of the FREDHOLM type, with the results of HILBERT and others, finds no place in these lectures. This phase of the subject constitutes an extended field, and demands more adequate treatment than could be given here.

The solution of functional equations of a certain type is first considered, and the result is afterwards used freely to unify the treatment of different problems. The method is essentially that of successive approximations, but may have some advantage over the usual form in directness of application. It is applied in § 2 to prove the fundamental existence theorem for the ordinary linear differential equation of second order; in § 5 to solve a new boundary value problem for the linear partial differential equation of hyperbolic type—a problem which includes as special cases a number of those previously studied; in § 6 to present NEUMANN's method of solution of DIRICHLET's problem, and in § 9 to discuss the boundary value problem for the equation $\Delta u = cu + f$ for a small region.

In § 8 the existence of doubly periodic solutions of the equation $\Delta u = f$ is treated. The method illustrates the construction and application of a GREEN's function for cases other than those well known in the potential theory.

The concluding sections contain a discussion of the ordinary linear differential equation of second order, which includes the first boundary value problem, the existence of normal functions for the equation $y'' + \lambda A y = 0$, and the expansion of an arbitrary function in term of normal functions. The properties of the normal functions as solutions of minimum problems are made prominent by the method which is employed.

## § 1. THE SOLUTION OF CERTAIN FUNCTIONAL EQUATIONS.*

Consider the equation

$$f = g + Sf, \tag{1}$$

where $g$ is a known continuous function for values of its arguments in a region $R$, and $S$ a linear operator, so that $S(u+v) = Su + Sv$. It will be shown that a continuous solution $f$ may be found in a simple manner if the operator $S$ is of a certain type — is namely *convergent* according to the following definition. Let $\phi$ be any function continuous in the region $R$ considered: *The operator $S$ will be called convergent if the following conditions are satisfied:*

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* The method presented in this section has been employed in several particular cases, probably for the first time by LIOUVILLE. See BÔCHER, *An Introduction to the Study of Integral Equations* (Cambridge Mathematical Tracts), 1909.

1°. *$S\phi$ is a continuous function in $R$.*

2°. *The infinite series $\phi + S\phi + S^2\phi + \cdots$ converges in $R$ and represents a continuous function in $R$.*

3°. *The result of the operation $S$ on the function represented by this series is equal to the result of the term by term operation on the series.*

Suppose that $S$ is convergent, and that a solution* $f$ of (1) exists. Then

$$f = g + Sf = g + S(g + Sf) = g + Sg + S^2f$$
$$= g + Sg + S^2g + S^3f,$$

the equation

$$f = g + Sg + S^2g + \cdots + S^ng + S^{n+1}f$$

being obtained after $n$ substitutions of $g + Sf$ for $f$. If $S$ is convergent, the limit for $n = \infty$ may be taken. It follows that if a solution of (1) exists it has the form

$$f = g + Sg + S^2g + S^3g + \cdots. \tag{2}$$

Conversely, if $S$ is a convergent operator the function defined by equation (2) is a solution of equation (1), for

$$Sf = Sg + S^2g + S^3g + \cdots = f - g.$$

To summarize: *If $S$ is a convergent operator, the equation*

$$f = g + Sg \tag{1}$$

*admits a unique continuous solution, given by the equation*

$$f = g + Sg + S^2g + S^3g + \cdots. \tag{2}$$

---

* Continuous solutions are alone considered.

A simple example is furnished by the equation*

$$(3) \qquad f(x) = g(x) + \int_a^x K(x, y) f(y) dy.$$

The functions $g$ and $K$ are assumed to be continuous for all values of the variables in an interval $R$, and a continuous solution $f$ is required. The operator $S$ is convergent. For if $\gamma$ be the maximum of the absolute value of a function $\phi$, and $\kappa$ the maximum of $|K|$ in $R$, then

$$|S^n \phi| \leqq \frac{|x - a|^n \kappa^n}{n!} \cdot \gamma.$$

The series (2) therefore converges uniformly, and since the operation $S$ consists of multiplication by the function $K$ and integration, it may be preformed term by term. The operation $S$ is therefore convergent and the series (2) gives the unique solution of the equation.

## § 2. FUNDAMENTAL EXISTENCE THEOREM FOR THE ORDINARY DIFFERENTIAL EQUATION.

The fundamental existence theorem for the differential equation

$$(1) \qquad y'' + p(x) y' + q(x) y = f(x)$$

may be proved in a simple manner by the aid of the method of the preceding section. The coefficients are supposed to be continuous in a closed interval $R$. Let $a$ be a point in this interval. Then the following theorem holds:

*There exists a unique solution*† *of the differential equation* (1) *in the interval $R$ which satisfies the initial conditions*

$$(2) \qquad y(a) = \alpha, \qquad y'(a) = \alpha',$$

*where $\alpha$ and $\alpha'$ are given constants.*

Let $y'' = v$. If equations (2) are satisfied, then

---

*This is an integral equation of a type considered by *Volterra*, see BÔCHER, loc. cit.

† By a solution in the interval $R$ is meant a function which satisfies the differential equation at all points of the interval $R$. This definition demands the existence of $y''$ at each point, and hence implies the continuity of $y'$.

$$y'(x) = \int_a^x v(\xi)d\xi + \alpha',$$

$$(3)\quad \begin{aligned} y(x) &= \int_a^x dx \int_a^x v(\xi)d\xi + \alpha'(x-a) + \alpha \\ &= \int_a^x (x-\xi)v(\xi)d\xi + \alpha'(x-a) + \alpha. \end{aligned}$$

If $v$ is any continuous function this equation defines a function $y$ continuous together with $y'$ and $y''$, which satisfies the initial conditions (2). The necessary and sufficient condition that this function be a solution of the differential equation (1) is the following functional equation for $v$, found by substituting the above values of $y$, $y'$, $y''$ in equation (1):

$$(4)\quad \begin{aligned} v(x) = f(x) - p(x)\alpha' - q(x)\left[\alpha'(x-a) + \alpha\right] \\ - \int_a^x \left[p(x) + (x-\xi)q(x)\right]v(\xi)d\xi. \end{aligned}$$

Now this equation has the form of equation (3), § 1. There exists, therefore, in the interval $R$, a unique solution $v(x)$ of equation (4) and hence a unique solution $y(x)$ of equations (1) and (2), which is defined in terms of $v$ by equation (3). This solution, if $\alpha$ and $\alpha'$ are regarded as arbitrary constants, is the "general solution" of equation (1).

If the homogeneous equation

$$(1_0)\quad y'' + py' + qy = 0$$

be solved, first under the conditions $y(a) = 0$, $y'(a) = 1$, and second under the conditions $y(a) = 1$, $y'(a) = 0$, two solutions $\eta_1$ and $\eta_2$ are obtained, which are linearly independent and are connected by the relation

$$\eta_1'\eta_2 - \eta_2'\eta_1 = e^{-\int_a^x p dx}.$$

The general solution of $(1_0)$ may then be written in the form

$$y = c_1\eta_1 + c_2\eta_2,$$

when $c_1$ and $c_2$ are arbitrary constants. The general solution of the non-homogeneous equation (1) is expressed in terms of these solutions of the homogeneous equation by the well-known form

$$y = c_1\eta_1 + c_2\eta_2 + \int_a^x [\eta_1(x)\eta_2(\xi) - \eta_1(\xi)\eta_2(x)]\, e^{\int_a^\xi p dx} f(\xi) d\xi$$

obtained by the method of variation of parameters.

If the coefficients in equation (1) are analytic at $x = a$, i. e., are expressible as power series in $(x - a)$, then the solution of the differential equation (1) under the initial conditions (2) may be obtained by the power series method as an analytic function. On account of the uniqueness of the solution it is evident that every solution of equation (1) is analytic if the coefficients are analytic.

## § 3. THE CAUCHY-KOWALEWSKI EXISTENCE THEOREM FOR THE PARTIAL DIFFERENTIAL EQUATION.

The general linear partial differential equation of second order is

$$(1) \quad Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0.$$

A problem analogous to that of the preceding section is to determine a solution $u$ of this equation such that along a given curve in the $(x, y)$ plane $u$ and $\partial u/\partial n$, where $n$ is the normal to the given curve, take on assigned values — in other words, to determine an integral surface $u = u(x, y)$ of the equation (1), which passes through a given space curve and is tangent along this curve to a given strip. In case the coefficients of the differential equation and all functions which enter into the initial conditions are analytic the power series method as used by Cauchy, and extended by Kowalewski, gives the following solution of the problem.*

*Analytic case: There exists a unique analytic solution of the problem, except in case the given curve in the $(x, y)$ plane (projection of the given space curve) satisfies the differential equation*

$$Ady^2 - 2Bdxdy + Cdx^2 = 0,$$

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* See, e. g., Hedrick, *On the characteristics of differential equations*, Annals of Mathematics, 2d series, 4 (1903), p. 145.

*i. e., except for curves* $\zeta(x, y) = const.$, *where* $\zeta$ *satisfies the equation*

$$(2) \qquad A\zeta_x^2 + 2B\zeta_x\zeta_y + C\zeta_y^2 = 0.$$

Can a solution of the differential equation (1) be found which satisfies the initial conditions if the functions involved in these conditions are not analytic? For some equations this may be done, as will be seen in § 5. On the other hand, for the equation

$$u_{xx} + u_{yy} = 0,$$

the values of $u$ and $\partial u/\partial n$ can not be assigned arbitrarily along a given curve.[*] There is therefore a real difference between the cases of analytic and non-analytic initial conditions, and the limitation of the Cauchy-Kowalewski existence theorem to the analytic case is not caused by the method of proof alone.

In the further developments no restriction as to the analytical character of the functions which occur will be assumed unless the fact is expressly stated. These functions will however be assumed to be continuous in the region under consideration.

## § 4. CLASSIFICATION OF THE PARTIAL DIFFERENTIAL EQUATIONS.[†]

Let the independent variables in the general equation

$$(1) \quad Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

be replaced by the new variables $\xi$, $\eta$, the equations of transformation being

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y).$$

Then

$$u_x = u_\xi\xi_x + u_\eta\eta_x,$$
$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx},$$

with similar equations for $u_y$, $u_{xy}$, $u_{yy}$. After substitution of these values equation (1) takes the form

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[*] This was pointed out by HADAMARD, Bulletin of Princeton University, April, 1902. So far as I know no general treatment of this question has been made.

[†] LAPLACE, *Œuvres*, 9, p. 21. DUBOIS-REYMOND, Crelle, 104 (1889), p. 241.

$$(2)\qquad \alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} + \delta u_{\xi} + \epsilon u_{\eta} + \phi u + \psi = 0,$$

where

$$\begin{aligned}
\alpha &= A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2,\\
\beta &= A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y,\\
\gamma &= A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2,\\
\delta &= A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y,\\
\epsilon &= A\eta_{xx} + 2B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y,\\
\phi &= F,\\
\psi &= G.
\end{aligned}$$

The discriminant $AC - B^2$ is a covariant, since

$$\alpha\gamma - \beta^2 = (\xi_x\eta_y - \eta_x\xi_y)^2(AC - B^2),$$

and equations of the form (1) may be classified according to its value.

If $AC - B^2 < 0$ in the region considered, the equation is said to belong to the *hyperbolic type.* In this case the first member of the equation

$$(3)\qquad A\zeta_x^2 + 2B\zeta_x\zeta_y + C\zeta_y^2 = 0$$

may be resolved into distinct real linear factors. Hence two independent functions $\xi(x, y)$, $\eta(x, y)$ may be found which satisfy this equation.* Choose these functions for $\xi$, $\eta$ in the equations of transformation. Then $\alpha = 0$, $\gamma = 0$. Now $0 > \alpha\gamma - \beta^2 = -\beta^2$ and therefore $\beta$ does not vanish. The equation (2) may then be divided by $2\beta$, the result being

$$u_{\xi\eta} + \frac{1}{2\beta}(\delta u_{\xi} + \epsilon u_{\eta} + \phi u + \psi) = 0.$$

On changing the notation this equation takes the form

$$u_{xy} = au_x + bu_y + cu + f,$$

*the normal form for equations of hyperbolic type.*

If $AC - B^2 = 0$ in the region considered, the equation (1) is said to belong to the *parabolic type.* In this case the first member of equation (3) is the square of a linear expression. One function $\eta(x, y)$ may be found which satisfies (3), but no other solution functionally independent of $\eta$. In the equations of transformation choose $\eta$ as this solution. The coefficient $\gamma$ is then zero. Now $\alpha\gamma - \beta^2 = 0$, and therefore $\beta = 0$ also. On changing the notation as before, the equation takes the form

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* According to the elementary theory of partial differential equations of the first order these solutions are obtained by the integration of ordinary differential equations. Their functional independence is readily seen as a consequence of the fact that $AC - B^2 \neq 0$.

$$u_{xx} = au_x + bu_y + cu + f,$$

*the normal form for equations of parabolic type.*

If $AC - B^2 > 0$, the equation belongs to the *elliptic type.* In this case two complex solutions $\phi$, $\psi$, of (3) are to be found, of the form

$$\phi = \xi + i\eta, \qquad \psi = \xi - i\eta.$$

Choose the functions $\xi$, $\eta$ to define the transformation. Now $\phi$ satisfies equation (3), that is,

$$A(\xi_x^{\,2} + 2i\xi_x\eta_x - \eta_x^{\,2}) + 2B[\xi_x\xi_y - \eta_x\eta_y + i(\xi_x\eta_y + \eta_x\xi_y)] \\ + C(\xi_y^{\,2} + 2i\xi_y\eta_y - \eta_y^{\,2}) = 0.$$

Therefore $\beta = 0$, and $\alpha = \gamma$. But $\alpha\gamma - \beta^2 = \alpha\gamma > 0$, so that $\alpha \neq 0$. After dividing by $\alpha$ and changing the notation, the equation (2) takes the form

$$u_{xx} + u_{yy} = au_x + bu_y + cu + f,$$

*the normal form for equations of the elliptic type.*

It is evident that this classification applies as well to equations of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y),$$

which are not linear with respect to $u$, $u_x$, $u_y$. In an important memoir on the subject of the analytical character of solutions of partial differential equations, SERGE BERNSTEIN[*] has made a similar classification of the general non-linear differential equation of second order,

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0,$$

based upon the value of the determinant

$$4\frac{\partial F}{\partial u_{xx}}\frac{\partial F}{\partial u_{yy}} - \left(\frac{\partial F}{\partial u_{xy}}\right)^2.$$

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[*] Mathematische Annalen, 59 (1904), p. 20.

## § 5. THE BOUNDARY VALUE PROBLEMS OF THE EQUATIONS OF HYPERBOLIC TYPE.

The different boundary value problems which arise in connection with the partial differential equation of hyperbolic type,

$$u_{xy} = au_x + bu_y + cu + f, \tag{1}$$

have been treated by distinct methods, based for the most part on successive approximations and on RIEMANN's method.* A new boundary value problem will be treated in this section, which includes as special cases many of those previously studied.

Consider a rectangle $R$ containing a point $(x_0, y_0)$ and bounded by the lines

$$x = x_1, \quad y = y_1, \quad x = x_2, \quad y = y_2.$$

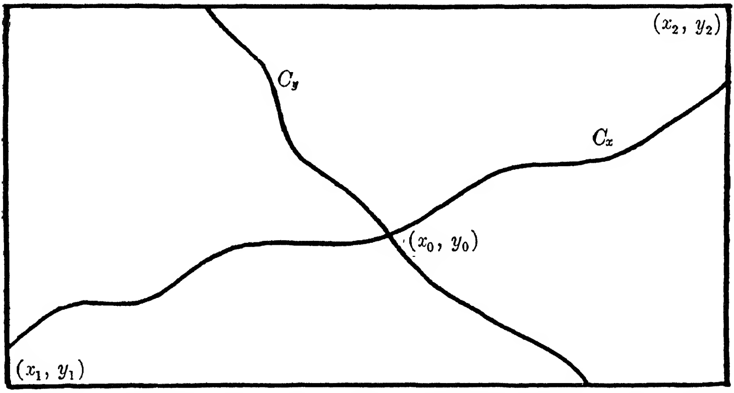
Let $C_x$ and $C_y$ be two curves intersecting at $(x_0, y_0)$, representable in the form

$$C_x: \quad y = \phi(x), \qquad C_y: \quad x = \psi(y),$$

where it is assumed that for $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$, the functions $\phi(x)$ and $\psi(y)$ are continuous and single-valued, that the derivative $\phi'(x)$ is continuous, and that the inequalities

$$y_1 \leqq \phi(x) \leqq y_2, \qquad x_1 \leqq \psi(y) \leqq x_2 \tag{2}$$

are satisfied.



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*See for example SOMMERFELD, loc. cit., and R. D'ADÉMAR, *Les equations aux dérivées partielles à caractéristiques réelles*, Paris, 1907.

*It is required to determine a solution* of the equation*

$$(1) \qquad u_{xy} = au_x + bu_y + cu + f$$

*in $R$, which satisfies the conditions*

$$(3) \qquad (u)_{y=\phi(x)} = U(x), \qquad (u_y)_{x=\psi(y)} = Y(y).$$

The coefficients of equation (1) are supposed continuous in $R$, $U(x)$, $Y(y)$ are given functions continuous in the intervals $(x_1, x_2)$ and $(y_1, y_2)$ respectively, and the derivative $U'(x)$ is continuous in $(x_1, x_2)$.

Let $u_{xy} = v$. Then if $u$ satisfies the conditions (3),

$$u_y = \int_{\psi(y)}^{x} v(\xi, y)d\xi + Y(y),$$

$$u = \int_{\phi(x)}^{y}\int_{\psi(\eta)}^{x} v(\xi, \eta)d\xi d\eta + \int_{\phi(x)}^{y} Y(\eta)d\eta + U(x),$$

$$u_x = -\phi'(x)\int_{\psi[\phi(x)]}^{x} v[\xi, \phi(x)]\,d\xi + \int_{\phi(x)}^{y} v(x, \eta)d\eta - \phi'(x)\,Y[\phi(x)] + U'(x).$$

If $v$ is any continuous function in $R$, the function $u$, defined by the second of the above equations, is continuous in $R$ together with the derivatives $u_x$, $u_y$, $u_{xy}$, and satisfies the conditions (3). The necessary and sufficient condition that $u$ satisfies the differential equation (1) is the following functional equation for $v$, found by substituting the above expressions for $u$, $u_x$, $u_y$, $u_{xy}$ in (1):

$$v(x, y) = g(x, y) + a(x, y)\int_{\phi(x)}^{y} v(x, \eta)d\eta$$

$$(4) \qquad - a(x, y)\phi'(x)\int_{\psi[\phi(x)]}^{x} v[\xi, \phi(x)]d\xi + b(x, y)\int_{\psi(y)}^{x} v(\xi, y)d\xi$$

$$+ c(x, y)\int_{\phi(x)}^{y}\int_{\psi(\eta)}^{x} v(\xi, \eta)d\xi d\eta,$$

---

* I. e., a function $u$ which satisfies (1) at each point of $R$. This assumes the existence of $u_{xy}$ at each point, and hence implies the continuity of $u$, $u_x$, $u_y$ in $R$.

where

$$g(x, y) = a(x, y)\{U'(x) - \phi'(x)Y[\phi(x)]\} + b(x, y)Y(y)$$

(4′)

$$+ c(x, y)\left[\int_{\phi(x)}^{y} Y(\eta)d\eta + U(x)\right] + f(x, y).$$

This equation has the form

$$v = g + Sv,$$

where $g$ is a known continuous function and $S$ a linear operator. There exists a unique solution

$$v = g + Sg + S^2g + \cdots$$

if the operator $S$ is convergent, as defined in § 1, i. e., if the series

(5) $$z + Sz + S^2z + \cdots,$$

formed for any function $z$ which is continuous in $R$, converges uniformly, and if the operation $S$ may be preformed on this series term by term. From the definition of the operator $S$ by equation (4) it is seen at once that

$$\lim_{n=\infty} SR_n = 0,$$

if $R_n$ approaches zero uniformly in $R$ as $n$ increases indefinitely. It follows that the operation $S$ may be preformed term by term on the series, if the convergence is uniform. The operator $S$ is therefore convergent, and a unique solution of equation (4) exists, if the series (5) converges uniformly. That this is the fact, will be shown by forming a "majorant operator" to $S$, i. e., an operator which gives rise to a series of the form (5), each term of which is positive and greater than the absolute value of the corresponding term of (5). This new series will be shown to be uniformly convergent.

Let $\gamma$ be the greatest of the maxima of

$$|a|, \ 2|a\phi'|, \ 2|b|, \ \sqrt{|c|}$$

in $R$, and write

$$\alpha = x_1 - (x_2 - x_1), \qquad \beta = y_1 - (y_2 - y_1).$$

Then it follows on account of the inequalities (2), that for all $x$, $y$ in $R$,

$$|x-\psi(y)| \leqq x_2 - x_1 \leqq x - \alpha,$$
$$|y-\phi(x)| \leqq y_2 - y_1 \leqq y - \beta.$$

Define three operators $S_1$, $S_2$, $S_3$ by the equations

$$S_1 w = \gamma \int_\alpha^x w(\xi, y)d\xi, \qquad S_2 w = \gamma \int_\beta^y w(x, \eta)d\eta,$$
$$S_3 w = \gamma^2 \int_\beta^y \int_\alpha^x w(\xi, \eta)d\xi d\eta,$$

where $w(x, y)$ is a function defined for all values of $x$, $y$ in the intervals $(\alpha, x_2)$, $(\beta, y_2)$. Let $z(x, y)$ be any function continuous in $R$, and suppose that $w(x, y) \geqq 0$ in its region of definition, and that

$$w(\alpha + |x'|, \beta + |y'|) \geqq |z(x, y)| = |z[\psi(y) + x', \phi(x) + y']|,$$

where $x' = x - \psi(y)$, $y' = y - \phi(x)$, for all values of $x$, $y$, in $R$. Then the functions $(S_1 + S_2 + S_3)w$ and $Sz$ satisfy this same inequality,* i. e.,

$$[(S_1 + S_2 + S_3)w]_{\substack{\alpha+|x'|\\ \beta+|y'|}} \geqq |Sz|.$$

It follows, therefore, that

$$[(S_1 + S_2 + S_3)^k w]_{\substack{\alpha+|x'|\\ \beta+|y'|}} \geqq |S^k z|.$$

Since $w \geqq 0$ the function $(S_1 + S_2 + S_3)^k w$ is an increasing func-

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* The proof of this fact is obtained at once by comparing the values of the integrals which occur in $Sz$ with the integrals of $S_1 w$, $S_2 w$, $S_3 w$; for example, the integrals

$$\int_{\psi(y)}^x z(\xi, y)dy \quad \int_{\psi(y)}^{\psi(y)+x'} z(\xi, y)d\xi = \int_0^{x'} z[\psi(y) + x', y]dx'$$

and

$$\int_\alpha^{\alpha+|x'|} w(\xi, \beta + |y'|)d\xi \quad \int_0^{|x'|} w(\alpha + |x'|, \beta + |y'|)dx'.$$

The consideration of the second integral in $Sz$ is reduced to the one above, since

$$\int_{\psi[\phi(x)]}^x z[\xi, \phi(x)]d\xi = \left[\int_{\psi(y)}^x z(\xi, y)d\xi\right]_{y=\phi(x)}.$$

tion of both $x$ and $y$, and therefore

$$(S_1 + S_2 + S_3)^k w \geqq |S^k z|$$

for all points $(x, y)$ in $R$. Choose for $w$ the maximum $\delta$ of $|z|$ in $R$. Then the following statement is justified :

*The series* (5) *converges absolutely and uniformly in $R$, if the series of positive terms*

$$(6) \qquad \sum_{k=0}^{\infty} (S_1 + S_2 + S_3)^k \delta$$

*converges uniformly in $R$.*

The operators $S_1$, $S_2$, $S_3$ are commutative and associative, so that in the series (6) each term of the form $S_1^{k_1} S_2^{k_2} S_3^{k_3} \delta$ occurs with the factor

$$\frac{(k_1 + k_2 + k_3)!}{k_1!\, k_2!\, k_3!} .$$

Now $S_3 w = S_1 S_2 w = S_2 S_1 w$. Then

$$\sum_{k=0} (S_1 + S_2 + S_3)^k \delta = \sum_{m,\, n=0}^{\infty} A_{m,\, n} S_1^{\,m} S_2^{\,n} \delta,$$

where

$$A_{m,\, n} = \sum \frac{(k_1 + k_2 + k_3)!}{k_1!\, k_2!\, k_3!} ,$$

the summation being extended over all positive integers (including zero) which satisfy the equations

$$k_1 + k_3 = m, \qquad k_2 + k_3 = n.$$

$A_{m,\, n}$ is increased if these two conditions be replaced by the single equation

$$k_1 + k_2 + 2k_3 = m + n.$$

Now $A_{m,\, n}$ is increased if each value of $k_3$ is replaced by a greater value, and is therefore increased if the above equation be replaced by

$$k_1 + k_2 + k_3 = m + n.$$

The summation under the last condition gives

$$(1 + 1 + 1)^{m+n} = 3^{m+n}.$$

Therefore

$$A_{m,\, n} < 3^{m+n}.$$

Now

$$S_1^m S_2^n \delta = \frac{\gamma^m (x-\alpha)^m}{m\,!} \cdot \frac{\gamma^n (y-\beta)^n}{n\,!}\,\delta,$$

and therefore the terms of the series

$$\sum_{m,\,n=0}^{\infty} A_{m,\,n} S_1^m S_2^n \delta$$

are less than the terms of the series

$$\sum_{m,\,n=0}^{\infty} \frac{[3\gamma(x-\alpha)]^m [3\gamma(y-\beta)]^n}{m\,!\,n\,!}\,\delta = \delta e^{3\gamma(x-\alpha+y-\beta)}.$$

*The series* (6) *and* (5) *therefore converge uniformly and absolutely in* $R$, *and the operator* $S$ *of equation* (4) *is convergent.* The following existence theorem is therefore proved :

*There exists a unique solution of the differential equation*

$$u_{xy} = au_x + bu_y + cu + f$$

*in the rectangle* $R$, *which satisfies the conditions*

$$(u)_{y=\phi(x)} = U(x), \qquad (u_y)_{x=\psi(y)} = Y(y).$$

*This solution is given by the equations*

$$u = \int_{\phi(x)}^{y}\int_{\psi(\eta)}^{x} v(\xi,\,\eta)\,d\xi\,d\eta + \int_{\phi(x)}^{y} Y(\eta)\,d\eta + U(x),$$

$$v = g + Sg + S^2 g + \cdots,$$

*where* $g$ *is given by equation* (4′) *and* $S$ *is the operator which occurs in equation* (4).

By specializing the boundary conditions in this problem solutions are obtained for various boundary problems which have recieved special attention in the past. The following are examples :

1°. *The value of* $u$ *is given on the intersecting curves* $y = \phi(x)$, $x = x_0$ :

$$(u)_{y=\phi(x)} = U(x), \qquad (u)_{x=x_0} = V(y).$$

The second condition may be replaced by the equation

$$(u_y)_{x=x_0} = V'(y),$$

since the value of $u$ is known at one point of the line $x = x_0$, namely at the intersection of $y = \phi(x)$ with $x = x_0$. The problem[*] is therefore included in the general case for $\psi(y) = x_0$, $Y(y) = V(y)$.

2°. *The value of $u$ is given on the intersecting lines $x = x_0$, $y = y_0$.* This classical problem is included in 1° for $\phi(x) = y_0$. The first rigorous solution of this problem was obtained by PICARD[†] by the method of successive approximations.

3°. *The value of $u$ and of its normal derivative are given along a monotonic curve $C$.* Since the curve is monotonic these conditions are equivalent to giving $u$ and $u_x$ along the curve. By letting the curves $C_1$ and $C_2$ coincide in the monotonic curve $C$ the general problem is reduced to this important special case. The formulas are simplified by the fact that $\phi(x)$ and $\psi(y)$ are here inverse functions. This problem was also solved by PICARD.[‡]

Combinations of these special cases may be made to solve further boundary problems; for example HADAMARD's "mixed problem,"[§] is obtained by combining 1° and 3°.

## § 6. THE POTENTIAL EQUATION.

The potential equation

$$\Delta u = u_{xx} + u_{yy} = 0 \tag{1}$$

is the most important as well as the simplest differential equation of elliptic type. The solution of the boundary value problems for this equation play an important part in the treatment of the similar problems for the general equation of elliptic type.

Dirichlet's problem consists in determining a solution of the potential equation within a given region $R$ bounded by a closed curve $C$, which assumes given values on the boundary $C$.

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[*] First solved by PICARD for the special case [illegible]. See Note I in DARBOUX, *Théorie générale des surfaces*, 4, p. 353. More general problems of this nature have recently formed the subject of investigations by GOURSAT. See Annales de la Fac. de Toulouse, 2e série, t. V and VI.

[†] Note I in DARBOUX, *Théorie générale des surfaces*, 4, p. 353.

[‡] Journal de Math., 4e série, 6 (1890).

[§] Bulletin de la Soc. Math. de France, 1900, 1903.

Various methods have been devised for the solution of the problem, the simplest of which, in case the region $R$ is convex, is that of NEUMANN.[*] The results of § 1 are used in the following presentation of Neumann's method.

The region $R$ is bounded by the closed curve $C$

$$x = \xi(t), \qquad y = \eta(t),$$

where $t$ is the length of arc and varies from 0 to $l$. It will be assumed for simplicity that the functions $\xi$ and $\eta$ have continuous first and second derivatives.[†] The function

$$u(x, y) = \frac{1}{\pi}\int_0^l f(t)\,\frac{\partial\,\theta\,[x, y, \xi(t), \eta(t)]}{\partial t}\,dt,$$

where

$$\theta(x, y, \xi, \eta) = \text{arc tan}\,\frac{y-\eta}{x-\xi},$$

is a solution of $\Delta u = 0$ at all points not on the boundary, whatever be $f$. This may be verified immediately, since at all points not on $C$ the integrand is continuous together with all its derivatives, and the differentiation may be preformed under the sign of integration. The function $u$ is a "potential of a double distribution." As the point $(x, y)$ in the interior of $R$ approaches a point on the boundary, corresponding to the parameter value $t = s$, the value of $u$ approaches a definite value $u_{is}$ independent of the direction of approach. These values $u_{is}$ form a continuous function of $s$, and satisfy the equation[‡]

$$u_{is} = \frac{1}{\pi}\int_0^l f(t)\,\frac{\partial\,\theta(s, t)}{\partial t}\,dt + f(s),$$

where

$$\theta(s, t) = \theta\,[\,\xi(s), \eta(s), \xi(t), \eta(t)].$$

The necessary and sufficient condition that the function $u$ assume the given continuous boundary values $g(s)$ is therefore the following functional equation for $f$:

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[*] Leipziger Berichte, 1870, p. 50.

[†] It may be readily seen from this assumption that the function $\partial\theta(s, t)/\partial t$ which occurs below is continuous.

[‡] See e. g., C. NEUMANN, *Untersuchungen über das logarithmische und Newton'sche Potential*, Leipzig, 1877, p. 139.

$$f(s) = g(s) - \frac{1}{\pi}\int_0^l f(t)\frac{\partial\,\theta(s,\,t)}{\partial t}\,dt. \tag{2}$$

On account of the fact that

$$f(s) = \frac{f(s)}{\pi}\int_0^l \frac{\partial\,\theta(s,\,t)}{\partial t}\,dt = \frac{1}{\pi}\int_0^l f(s)\frac{\partial\,\theta(s,\,t)}{\partial t}\,dt,$$

this equation may be written in the form

$$f(s) = \frac{g(s)}{2} + \frac{1}{2\pi}\int_0^l [f(s) - f(t)]\frac{\partial\,\theta(s,\,t)}{\partial t}\,dt \tag{3}$$

or

$$f = \frac{g}{2} + Sf,$$

where $S$ is a linear operator, and is of the type considered in § 1. The operator $S$ is in this case convergent, according to the definition of § 1, if the series

$$\phi + S\phi + S'\phi + \cdots,$$

where $\phi$ is any function of $s$ continuous in the interval $(0,\,l)$, is uniformly convergent. From the form of $S$ it is seen that each term in this series depends only on the *variation* of the preceding term. If $m$ and $M$ are the minimum and maximum respectively of $\phi$, then

$$|\,S\phi\,| < \frac{M - m}{2}.$$

Consider now the function

$$\int_0^l \phi(t)\frac{\partial\,\theta(s,\,t)}{\partial t}\,dt.$$

Following Neumann, let $\alpha$ be that part of the interval $(0,\,l)$ within which $\phi$ lies between its minimum $m$ and its mean value $(M + m)/2$, and let $\beta$ be the remainder of the interval $(0,\,l)$. Write

$$\int_{(\alpha)}\frac{\partial\,\theta(s,\,t)}{\partial t}\,dt = \theta_s^{\alpha}, \qquad \int_{(\beta)}\frac{\partial\,\theta(s,\,t)}{\partial t}\,dt = \theta_s^{\beta}.$$

$\theta_s^\alpha$ is the sum of the angles subtended at the point $s$ of the curve by those portions of the curve which belong to $\alpha$, and similarly for $\theta_s^\beta$. Furthermore

$$\theta_s^\alpha + \theta_s^\beta = \pi.$$

Since the region is assumed to be convex, $\partial\theta/\partial t \geqq 0$, and therefore:

$$\frac{1}{\pi}\int_0^l \phi(t)\frac{\partial\theta(s,\,t)}{\partial t}\,dt \leqq \frac{m+M}{2\pi}\,\theta_s^\alpha + \frac{M}{\pi}\,\theta_s^\beta = M - \frac{M-m}{2\pi}\,\theta_s^\alpha,$$

$$\frac{1}{\pi}\int_0^l \phi(t)\frac{\partial\theta(s',\,t)}{\partial t}\,dt \geqq \frac{m}{\pi}\,\theta_{s'}^\alpha + \frac{m+M}{2\pi}\,\theta_{s'}^\beta = m + \frac{M-m}{2\pi}\,\theta_{s'}^\beta.$$

Therefore

$$(4)\qquad \frac{1}{\pi}\int_0^l \phi(t)\frac{\partial\theta(s,\,t)}{\partial t}\,dt - \frac{1}{\pi}\int_0^l \phi(t)\frac{\partial\theta(s',\,t)}{\partial t}\,dt \leqq (M-m)\left(1 - \frac{\theta_s^\alpha + \theta_{s'}^\beta}{2\pi}\right).$$

Now the quantity $\theta_s^\alpha + \theta_{s'}^\beta$ is always included between 0 and 1. In fact it may easily be shown * that there exists a fixed positive number $\rho$, such that for all points $s$ and $s'$ on the boundary, and for any division of the interval $(0,\,l)$ into the parts $\alpha$, $\beta$

$$(5)\qquad 1 - \frac{\theta_s^\alpha + \theta_s^\beta}{2\pi} \leqq \rho < 1.$$

Denote by $\delta\phi$ the variation of $\phi$ in the interval $(0,\,l)$, i. e., the difference between its maximum and its minimum: then, from (4) and (5),

$$(6)\qquad \delta\,\frac{1}{\pi}\int_0^l \phi(t)\frac{\partial\theta(s,\,t)}{\partial t}\,dt \leqq \rho\cdot\delta\phi.$$

Now

$$S\phi = \frac{1}{2\pi}\int_0^l \{\phi(s) - \phi(t)\}\frac{\partial\theta(s,\,t)}{\partial t}\,dt,$$

and

$$\delta\,\frac{\phi(s) - \phi(t)}{2} = \delta\phi.$$

Then

$$\delta S\phi < \rho \cdot \delta\phi.$$

Furthermore, as has been seen,

$$|S^n\phi| < \delta S^{n-1}\phi,$$

so that

$$|S^n\phi| < \rho^{n-1} \cdot \delta\phi.$$

Since $\rho < 1$ it follows that the series in question converges uniformly, and that $S$ is a convergent operator. The following theorems may therefore be stated:

*There exists a unique solution*

$$f = \tfrac{1}{2}\{g + Sg + S^2 g + \cdots\}$$

*of the equation*

$$f(s) = \frac{g(s)}{2} + Sf \equiv \frac{g(s)}{2} + \frac{1}{2\pi}\int_0^l \{f(s) - f(t)\}\frac{\partial\,\theta(s,\,t)}{\partial t}\,dt.$$

*The function*

$$u(x,\,y) = \frac{1}{\pi}\int_0^l f(t)\,\frac{\partial\,\theta\,[x,\,y,\,\xi(t),\,\eta(t)]}{\partial t}\,dt,$$

*satisfies the potential equation in the region $R$ and assumes the values $g(s)$ on the boundary of $R$.*

A solution of Dirichlet's problem is thus obtained for a convex region, when the assigned boundary values $g(s)$ are continuous.

It was pointed out by SCHWARZ that if the function $g(s)$ is in general continuous, but suffers a finite number of finite discontinuities, the problem may easily be reduced to the previous case. Suppose, for example, the function $g$ has the discontinuity $d$ at the point $(\alpha,\,\beta)$ of the boundary, but is elsewhere continuous. Now the function

$$\frac{d}{\pi}\arctan\frac{y-\beta}{x-\alpha}$$

is a solution of the potential equation within $R$ and assumes values on the boundary which are continuous except at $(\alpha,\,\beta)$ where they have the discontinuity $d$. Therefore, if $v(x,\,y)$ is the solution of Dirichlet's problem for the continuous boundary values

$$g(s) - \frac{d}{\pi} \arctan \frac{\eta(s) - \beta}{\xi(s) - \alpha};$$

the required solution of the original problem is

$$u(x, y) = v(x, y) + \frac{d}{\pi} \arctan \frac{y - \beta}{x - \alpha}.$$

As the point $(x, y)$ approaches the point $(\alpha, \beta)$ along the boundary, from one or the other side, the function $u(x, y)$ approaches one or the other of the values

$$v(\alpha, \beta) + \frac{d}{\pi}\theta, \qquad v(\alpha, \beta) + \frac{d}{\pi}(\theta - \pi),$$

where $\theta$ is the angle which the tangent to the curve at $(\alpha, \beta)$ makes with the $x$-axis. If $(x, y)$ approaches $(\alpha, \beta)$ from an interior point, the limiting value of $v$ depends on the direction $\theta'$ of approach, and is equal to

$$v(\alpha, \beta) + \frac{d}{\pi}\theta',$$

a value which is included between the limiting values approached along the boundary.

The fact last stated plays an important part in the method of alternation of SCHWARZ,[*] which makes it possible to pass from convex regions to those of more general character. *If Dirichlet's problem can be solved for two regions $R$, $R'$ having a part in common, it can be solved for the combined region $R + R'$.*

Suppose the region considered can be decomposed into two regions bounded by the convex curves $a\alpha$, $b\beta$. The values of $u$ are given on $a$ and $b$. Form the solution $u_1$ of $\Delta u = 0$ which assumes the given values on $a$, and values on $\alpha$ which join continuously onto the given values at $m$ and $n$. This function will have certain values $(u_1)_\beta$ on $\beta$. Form the solution $v_1$ of $\Delta u = 0$ which assumes the given values on $b$, and the values $(u_1)_\beta$ on $\beta$. This function will take on certain values

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[*] See his *Gesammelte Abhandlungen*, vol. 2, p. 157.

on $\alpha$. Form the solution $u_2$ of $\Delta u = 0$, which assumes
ven values on $a$, and the values $(v_1)_\alpha$ on $\alpha$, etc. Two
of functions $u_n$, $v_n$ are thus obtained. The proof is com-
l by showing that $u_n$ and $v_n$ approach definite limits, as $n$
ses indefinitely, and that these limits are solutions of
0, which coincide in the area bounded by $\alpha$ and $\beta$, and
e the given boundary values on $a$ and $b$. The solution
richlet's problem is thus obtained for the combined region.
sely allied to NEUMANN's method of solution of
CHLET's problem are those of KIRCHOFF,* ROBIN,† and
LOFF.‡ In the method of POINCARÉ § a series of func-
is formed, each of which satisfies the boundary condition,
hose limit satisfies the differential equation as well. The
recent method is that of FREDHOLM,‖ whereby the
ional equation (2) of this section is solved by a new
od. This as well as the older methods has been applied
lve the other linear boundary problems of the potential
y.¶

## § 7. THE EQUATION $\Delta u = f$; GREEN'S FUNCTIONS.

account of the existence of a solution of Dirichlet's prob-
here exists a function $g$ which satisfies the potential equa-
$\Delta u = 0$ at all points within the region $R$ and assumes the
s

$$-\log\frac{1}{\rho_{\xi\eta}^{xy}} = \log\sqrt{(x-\xi)^2+(y-\eta)^2}$$

e boundary $C$, where $(\xi, \eta)$ is a point in the interior of
The function

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cta Mathematica, 14 (1890), p. 180.

omptu Rendus, 104 (1887), p. 1834.

id., 125 (1897), p. 1026.

e his *Théorie du potential Newtonien*, Paris, 1899, p. 260.

versigt of Kongl. Vetenskaps Akad. Förhandlinger, 1900.

r the application of the FREDHOLM method see in particular PLEMELJ,
tshefte für Mathematik und Physik, 15 (1904), p. 337, and 18
, p. 180. For the older methods see, in particular, KORN, *Abhand-*
*zur Potentialtheorie*, Berlin, 1901.

$$G(x, y, \xi, \eta) = \log \frac{1}{\rho_{\xi\eta}^{xy}} + g(x, y, \xi, \eta)$$

is called the *Green's function* for the region $R$. It satisfies the equation $\Delta G = 0$ at all points of $R$ except the point $(\xi, \eta)$ and vanishes identically in $(\xi, \eta)$ when the point $(x, y)$ lies on the boundary $C$. This function is symmetrical, that is,

$$G(x, y, \xi, \eta) = G(\xi, \eta, x, y).$$

This is shown by the aid of Green's theorem,

$$(1) \qquad \int\int_R (v\Delta u - u\Delta v)dxdy = \int_C \left( v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds,$$

where $n$ is the direction of the normal to the curve $C$, taken as positive when pointing outward. In this equation take $u = G(x, y, \xi, \eta)$ and $v = G(x, y, \xi', \eta')$, and apply to the region formed by excluding from $R$ the points $(\xi, \eta)$ and $(\xi', \eta')$ by small circles. In the limit, when the radii of these circles approach zero, the equation

$$G(\xi, \eta, \xi', \eta') = G(\xi', \eta', \xi, \eta)$$

is obtained.

By a similar process an integral representation of the solution of the equation

$$(2) \qquad \Delta u = f(x, y)$$

which vanishes on the boundary $C$ is derived. The solution $u$ is substituted in equation (1), and the Green's function $G(x, y, \xi, \eta)$ is taken for $v$. Equation (1) is applied to the region formed by excluding from $R$ the point $(\xi, \eta)$ by a small circle. In the limit, when the radius of the circle approaches zero, the equation

$$u(\xi, \eta) = -\frac{1}{2\pi}\int\int_R G(x, y, \xi, \eta) f(x, y) dxdy$$

is obtained, or, since $G$ is symmetrical,

$$(3) \qquad u(x, y) = -\frac{1}{2\pi}\int\int_R G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta.$$

Conversely,* the function $u$ defined by equation (3) vanishes on the boundary: It is continuous together with its first derivatives $u_x$, $u_y$ in the interior of $R$ if $f$ is continuous, and it possesses second derivatives if the function $f$ satisfies the following condition of so-called "regular" continuity; there exist, for each point $(x, y)$ of $R$, a pair of positive constants $A$, $\lambda$ independent of $\Delta x$ and $\Delta y$, such that for all values of $\Delta x$ are $\Delta y$ sufficiently small,

$$(4) \quad |f(x+\Delta x, y+\Delta y) - f(x, y)| \leqq A\left[(\Delta x)^2 + (\Delta y)^2\right]^{\lambda}.$$

If this condition is satisfied the second derivatives of $u$ are continuous and satisfy the equation (2). The following theorem is an immediate consequence:

*The solution of the equation*

$$(2) \qquad \Delta u = f,$$

*where $f$ satisfies condition* (4), *which assumes given values on the boundary, is given by the equation*

$$(5) \quad u(x, y) = u_0(x, y) - \frac{1}{2\pi}\int\int_{(R)} G(x, y, \xi, \eta) f(\xi, \eta) d\xi d\eta,$$

*where $u_0(x, y)$ is the solution of $\Delta u = 0$ which assumes the given boundary values.*

## § 8. DOUBLY PERIODIC SOLUTIONS OF THE EQUATION $\Delta u = f$.†

The only doubly periodic solution ‡ of the potential equation, $\Delta u = 0$, is $u = c$, where $c$ is a constant. For if $u$ were such a solution, and $v$ the conjugate potential to $u$, then $u + iv$ would be a complex analytic function which has a value under a fixed finite limit for all values of $x$, $y$. But, as is well known, such a function is necessarily a constant. It will be shown, however, that the equation

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* The following statements regarding the first derivatives of $u$ are proved in any treatise on potential theory. The statement regarding the second derivatives is generally proved under the assumption that the function $f$ has continuous first derivatives. The statement in the form given here was proved by HÖLDER (*Dissertation*, Tübingen, 1882).

† This section is a reprint, with minor changes, of an article by the writer in the Transactions of the American Mathematical Society, 6 (1905), p. 159.

‡ By a doubly periodic solution is meant a doubly periodic function which is a solution of the equation in the period rectangle, and therefore in the entire plane. Such a function, in particular, has a value less than a fixed finite number for all values of $x$, $y$.

$$\Delta u = f(x, y), \tag{1}$$

where $f(x, y)$ is periodic in $x$ and in $y$ with the periods $a$ and $b$ respectively, has periodic solutions if $f$ satisfies a certain integral condition. A "doubly periodic GREEN's function," $G$, will be formed from known functions, and the desired solution of (1) found by quadrature from $G$ and $f$. The function $f$ is supposed to be regularly continuous, that is, to satisfy condition (4) of the preceding section.

Let $\Re$ denote the real part of the term before which it is written, and consider the function

$$\Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)}; \quad z = x + iy, \; \zeta = \xi + i\eta, \; \gamma = \alpha + i\beta,$$

where $\sigma(z)$ is the Sigma function of WEIERSTRASS, formed with the periods $a$, $ib$; and $\xi$, $\gamma$ are two points in the interior of the period rectangle $\Omega$ bounded by the lines $x = 0$, $y = a$, $x = 0$, $y = b$. This function is a solution of the potential equation within $\Omega$, except at the points $(\xi, \eta)$ and $(\alpha, \beta)$, and has the form

$$\log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + g,$$

where $g$ is a solution of the potential equation throughout $\Omega$.

Since for any integers $m$, $n$, the function $\sigma$ obeys the law

$$\sigma(z + ma + inb) = (-1)^{mn+m+n} e^{(m\eta_1 + n\eta_3)(2z + ma + inb)} \sigma(z),$$

where $\eta_1$, $\eta_3$ are certain complex constants,* we have

$$\Re \log \frac{\sigma(z + ma + inb - \zeta)}{\sigma(z + ma + inb - \gamma)} = \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} \tag{2}$$
$$- m\Re 2\eta_1(\zeta - \gamma) - n\Re 2\eta_3(\zeta - \gamma).$$

Define a real function $V$ by the equation

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* See e. g., BURKHARDT, *Elliptische Functionen*, p. 53.

$$V(x, y, \xi, \eta, \alpha, \beta) = \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} + \frac{x}{a} \Re 2\eta_1(\zeta-\gamma) + \frac{y}{b} \Re 2\eta_3(\zeta-\gamma).$$

Since the last two terms are linear in $x$ and $y$, $V$ has the form

$$V(x, y, \xi, \eta, \alpha, \beta) = \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + S(x, y, \xi, \eta, \alpha, \beta),$$

where $S$ is a known solution of

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0$$

within $\Omega$. Furthermore $V$ is doubly periodic in $x$, $y$ with the periods $a$, $b$ since from (2) the equation results:

$$\begin{aligned} V(x+ma, y+nb, \xi, \eta, \alpha, \beta) &= \Re \log \frac{\sigma(z+ma+inb-\zeta)}{\sigma(z+ma+inb-\gamma)} \\ &\quad + \frac{x+ma}{a} \Re 2\eta_1(\zeta-\gamma) + \frac{y+nb}{b} \Re 2\eta_3(\zeta-\gamma) \\ &= \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} + \frac{x}{a} \Re 2\eta_1(\zeta-\gamma) + \frac{y}{b} \Re 2\eta_3(\zeta-\gamma) \\ &= V(x, y, \xi, \eta, \alpha, \beta).^* \end{aligned}$$

*The function*

$$G(x, y, \xi, \eta, \alpha, \beta) = V(x, y, \xi, \eta, \alpha, \beta) - S(\alpha, \beta, \xi, \eta, a, \beta)$$

*will be called the doubly periodic Green's function for the periods $a$, $b$. This function has the following characteristics:*

1°. Except at $(\xi, \eta)$ and $(\alpha, \beta)$, $G$ is, within $\Omega$, a solution of the equation $\Delta u = 0$.

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[*] In the same manner may be formed a doubly periodic function

$$V(x, y, \xi_1, \eta_1, \cdots, \xi_n, \eta_n) = \sum_{i=1}^{n} c_i \log \sqrt{(x-\xi_i)^2 + (y-\eta_i)^2} + S(x, y, \xi_1, \eta_1, \cdots, \xi_n, \eta_n)$$

with any number of logarithmic singularities, where $S$ is a solution of LAPLACE's equation in $\Omega$ with respect to the variables $x$, $y$, provided that $c_1 + c_2 + \cdots + c_n = 0$.

2°. $G$ has the form

$$G = \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + R(x, y, \xi, \eta, \alpha, \beta),$$

where $R$ is a known function, which, with respect to the variables $x$, $y$, is a solution of LAPLACE's equation within $\Omega$, and which satisfies for all values of $\xi$, $\eta$ in $\Omega$, the equation

$$R(\alpha, \beta, \xi, \eta, \alpha, \beta) = 0.$$

3°. $G$ is doubly periodic in $x$, $y$ with the periods $a$, $b$.

*The functions $G$ and $R$ obey the following laws of reciprocity:*

$$G(x, y, \xi, \eta, \alpha, \beta) + \log \frac{1}{\sqrt{(x-\alpha)^2+(y-\beta)^2}} = G(\xi, \eta, x, y, \alpha, \beta) + \log \frac{1}{\sqrt{(\xi-\alpha)^2 + (\eta-\beta)^2}},$$

$$R(x, y, \xi, \eta, \alpha, \beta) = R(\xi, \eta, x, y, \alpha, \beta).$$

To prove these laws apply GREEN's theorem,

$$\int\int (v\Delta u - u\Delta v) dx\, dy = \int \left( v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds,$$

to the region $\Omega'$ formed by excluding from $\Omega$ the circles $c(\xi, \eta)$, $c(\xi', \eta')$, $c(\alpha, \beta)$ of radius $r$ about the points $(\xi, \eta)$, $(\xi', \eta')$, $(\alpha, \beta)$ respectively, and choose

$$u = G(x, y, \xi, \eta, \alpha, \beta), \qquad v = G(x, y, \xi', \eta', \alpha, \beta).$$

Since $u$ and $v$ are solutions of the potential equation within $\Omega'$, the double integral over $\Omega'$ is zero. Furthermore, since $u$ and $v$ are doubly periodic in $x$, $y$, each assumes equal values at opposite points of the bounding lines of the rectangle $\Omega$, while at these points the normal derivative of each assumes values numerically equal but opposite in sign. Therefore the line integral over the sides of the rectangle $\Omega$ is zero, and we have, replacing $ds$ by $rd\theta$,

$$\int_{c(\xi,\eta)} \frac{1}{r} G(x, y, \xi', \eta', \alpha, \beta) r d\theta - \int_{c(\xi',\eta')} \frac{1}{r} G(x, y, \xi, \eta, \alpha, \beta) r d\theta$$
$$+ \int_{c(\alpha,\beta)} \frac{1}{r} \{ G(x, y, \xi, \eta, \alpha, \beta) - G(x, y, \xi', \eta', \alpha, \beta) \} r d\theta + h = 0,$$

where

$$\lim_{r=0} h = 0.$$

But

$$G(x, y, \xi, \eta, \alpha, \beta) - G(x, y, \xi', \eta', \alpha, \beta)$$
$$= \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\xi')^2 + (y-\eta')^2}}$$
$$+ R(x, y, \xi, \eta, \alpha, \beta) - R(x, y, \xi', \eta', \alpha, \beta),$$

and

$$R(\alpha, \beta, \xi, \eta, \alpha, \beta) = 0, \qquad R(\alpha, \beta, \xi', \eta', \alpha, \beta) = 0.$$

We obtain therefore in the limit $r = 0$, by well known methods,

$$G(\xi, \eta, \xi', \eta', \alpha, \beta) - G(\xi', \eta', \xi, \eta, \alpha, \beta)$$
$$+ \log \frac{1}{\sqrt{(\alpha-\xi)^2 + (\beta-\eta)^2}} - \log \frac{1}{\sqrt{(\alpha-\xi')^2 + (\beta-\eta')^2}} = 0,$$

or writing $x, y$ for $\xi', \eta'$,

$$G(x, y, \xi, \eta, \alpha, \beta) + \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}}$$
$$= G(\xi, \eta, x, y, \alpha, \beta) + \log \frac{1}{\sqrt{(\xi-\alpha)^2 + (\eta-\beta)^2}}.$$

which is the law of reciprocity for $G$. If we replace $G$ in this equation by its expression from 2° we have immediately,

$$R(x, y, \xi, \eta, \alpha, \beta) = R(\xi, \eta, x, y, \alpha, \beta).$$

Thus $R$ is symmetrical with respect to $x, y$ and $\xi, \eta$ and is therefore a solution of

$$R_{\xi\xi} + R_{\eta\eta} = 0$$

within $\Omega$.

Suppose a doubly periodic solution $u$ of equation (1) exists, for the periods $a, b$, where $f$ is regularly continuous and doubly periodic with the same periods. If a second solution of the same nature existed, the difference of the two would be a doubly periodic solution of the potential equation, and therefore a constant. It follows that a doubly periodic solution of (1) for the periods $a$, $b$ is uniquely determined if its value at a fixed point is given.

Apply GREEN's theorem to the period rectangle $\Omega$, choosing for $u$ a doubly periodic solution of (1) and taking $v = 1$. Since the integral over the boundary vanishes, the following equation results:

$$\int_0^b \int_0^a \Delta u \, dx \, dy = \int_0^b \int_0^a f(x, y) dx \, dy = 0.$$

This equation is a necessary condition for the existence of a doubly periodic solution of (1). We shall now show that it is also sufficient. Consider the function

$$\begin{aligned} u(\xi, \eta) &= -\frac{1}{2\pi}\int_0^b \int_0^a G(x, y, \xi, \eta, \alpha, \beta) f(x, y) dx \, dy \\ &= -\frac{1}{2\pi}\int_0^b \int_0^a \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} f(x, y) dx \, dy \\ &\quad + \frac{1}{2\pi}\int_0^b \int_0^a \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} f(x, y) dx \, dy \\ &\quad - \frac{1}{2\pi}\int_0^b \int_0^a R(x, y, \xi, \eta, \alpha, \beta) dx \, dy. \end{aligned}$$

Since

$$R_{\xi\xi} + R_{\eta\eta} = 0,$$

we see at once, from the potential theory, that $u(\xi, \eta)$ is a solution of

$$u_{\xi\xi} + u_{\eta\eta} = f(\xi, \eta).$$

Furthermore, putting $\xi = \alpha$, $\eta = \beta$ we have

$$u(\alpha, \beta) = 0,$$

since

$$R(x, y, \alpha, \beta, \alpha, \beta) = R(\alpha, \beta, x, y, \alpha, \beta) = 0.$$

From the law of reciprocity for $G$ it follows that

$$G(x, y, \xi + ma, \eta + nb) - G(x, y, \xi, \eta, \alpha, \beta)$$

$$= \log \frac{1}{\sqrt{(\xi + ma - \alpha)^2 + (\eta + nb - \beta)^2}} - \log \frac{1}{\sqrt{(\xi - \alpha)^2 + (\eta - \beta)^2}},$$

and therefore

$$u(\xi + ma, \eta + nb) - u(\xi, \eta)$$

$$= -\frac{1}{2\pi} \log \sqrt{\frac{(\xi - \alpha)^2 + (\eta - \beta)^2}{(\xi + ma - \alpha)^2 + (\eta + nb - \beta)^2}} \int_0^b \int_0^a f(x, y) dx dy.$$

Therefore, if

$$\int_0^b \int_0^a f(x, y) dx\, dy = 0,$$

the function $u$ possesses the periods $a$, $b$. We have therefore proved the theorems:

*The necessary and sufficient condition for the existence of a doubly periodic solution (periods $a$, $b$) of the equation*

$$(1) \qquad \Delta u = f(x, y),$$

*where $f$ is a regularly continuous doubly periodic function with the periods $a$, $b$, is that $f$ satisfy the equation*

$$\int_0^b \int_0^a f(x, y) dx\, dy = 0.$$

*If this condition is satisfied, then the doubly periodic solution of* (1), *with periods $a$, $b$, which assumes the value $C$ at $x = \alpha$, $y = \beta$ is uniquely determined, and is given by the formula:*

$$u(\xi, \eta) = -\frac{1}{2\pi} \int_0^b \int_0^a G(x, y, \xi, \eta, \alpha, \beta) f(x, y) dx\, dy + C,$$

*where $G$ is a known function, expressible in terms of Sigma functions.*

## § 9. THE EQUATION $\Delta u = cu + f$.*

The coefficients $c$ and $f$ of the equation

$$(1) \qquad \Delta u = cu + f$$

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*SCHWARZ (1885), see his *Gesammelte Abhandlungen* 1, p. 241. PICARD, Journal de Mathématiques (ser. 4), 6 (1890), p. 145. The following presentation is based on the result of § 1 above.

are supposed to be regularly continuous, i. e., to satisfy condition (4) of § 7. Suppose a solution $u$ exists, within the region $R$, which assumes the values $g(s)$ on the boundary $C$ of $R$. Let $v$ be the solution of the equation

$$\Delta v = f$$

which assumes the values $g(s)$ on $C$. Then

$$\Delta(u - v) = cu,$$

and the function $u - v$ vanishes on the boundary. This function may therefore be represented in the form

$$u(x,\ y) - v(x,\ y) = -\frac{1}{2\pi}\int\int_R G(x,\ y,\ \xi,\ \eta)c(\xi,\ \eta)u(\xi,\ \eta)d\xi d\eta,$$

or

$$(2)\quad u(x,\ y) = v(x,\ y) - \frac{1}{2\pi}\int\int_R G(x,\ y,\ \xi,\ \eta)c(\xi,\ \eta)u(\xi,\ \eta)d\xi d\eta,$$

where $G$ is the Green's function of § 7. Conversely, a continuous solution $u$ of this functional equation takes on the boundary values $g(s)$, since the integral vanishes on the boundary, and it satisfies the differential equation (1) provided that $cu$ is regularly continuous. The function $v$ has continuous second derivatives inside of $R$, and $c$ is regularly continuous by assumption. Now the function

$$\int\int_R G(x,\ y,\ \xi,\ \eta)\phi(\xi,\ \eta)d\xi d\eta,$$

where $\phi$ is continuous, has continuous first derivatives and is therefore regularly continuous. It follows that a continuous solution $u$ of (2) has continuous first derivatives, and hence, since $cu$ will then be regularly continuous, that $u$ has continuous second derivatives $u_{xx}$, $u_{yy}$ which satisfy the equation

$$\Delta u = \Delta v + cu = cu + f.$$

*The solution of the boundary value problem of the differential equation* (1) *is equivalent to finding a continuous solution $u$ of the functional equation*

$$u = v + Su,$$

*here*

$$Su = -\frac{1}{2\pi}\int\int_R G(x, y, \xi, \eta)c(\xi, \eta)u(\xi, \eta)d\xi d\eta.$$

The operation $S$ may be preformed term by term on the
ries

$$\phi + S\phi + S^2\phi + \cdots,$$

here $\phi$ is any function continuous in $R$, if the series is uni-
rmly convergent; the operator will then be convergent ac-
rding to the definition of § 1, and there will exist a unique
lution of the boundary value problem.

Let the maxima of $|\phi|$ and $|c|$ in $R$ be $\delta$ and $\gamma$ respectively.
ow the Green's function is positively infinite at the point
$, \eta)$ and is zero on the boundary. It follows that $G$ is not
egative in $R$, for a solution of the potential equation in a
sed region [in this case the region formed by excluding the
oint $(x, y)$ from $R$ by a circle so small that $G$ is positive on
s circumference] has its maxima and minima on the boundary.
herefore

$$|S\phi| < \frac{\delta\gamma}{2\pi}\int\int_R G d\xi d\eta.$$

The convergence of the operator $S$ is a consequence of the
llowing lemma: *The region $R$ may be taken so small (in area)*
*at the value of the integral*

$$\int\int_R G d\xi d\eta$$

*less than a preassigned positive quantity $\epsilon$, however small.* Sup-
ose $G'$ is the Green's function for a region $R'$ lying entirely
ithin $R$. Then, since $G$ is positive in $R$, as has been seen,
d $G'$ vanishes on the boundary of $R'$, the difference $G - G'$,
hich satisfies $\Delta u = 0$ at all points within $R'$ is positive at all
ints of the boundary of $R'$. This difference is therefore posi-
e at all points of $R'$, since it cannot have a minimum in $R'$,
$G' < G$. That is, if $R$ is replaced by a region $R'$ lying
thin $R$ the value of the Green's function is decreased in each
int.

Now, since $G$ becomes only logarithmicaly infinite at $\xi = x$, $\eta = y$, the point $(x, y)$ may be included in a circle $k$, so small that

$$\int\int_k G d\xi d\eta < \frac{\epsilon}{2},$$

and on account of the above developed property of $G$, this inequality remains satisfied if the region $R$ is replaced by a region $R'$ lying within $R$. Outside this circle the function $G$ has a finite maximum $M$, which is decreased if $R$ be replaced by $R'$. Let the region $R$ be decreased in this manner. Then the area may be taken so small that

$$\int\int_{R-k} G d\xi d\eta < M \int\int_{R-k} d\xi d\eta < \frac{\epsilon}{2}.$$

The circle $k$, by this decrease in $R$, is to remain fixed. The lemma above stated is therefore proved.

Let this quantity $\epsilon$ be chosen so small that

$$\frac{\gamma\epsilon}{2\pi} < 1.$$

Then, since

$$|S\phi| < \frac{\gamma\epsilon}{2\pi}\delta,$$

$$|S^2\phi| < \left(\frac{\gamma\epsilon}{2\pi}\right)^2 \delta,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$|S^n\phi| < \left(\frac{\gamma\epsilon}{2\pi}\right)^n \delta,$$

it follows that the series

$$\phi + S\phi + S^2\phi + \cdots$$

is uniformly convergent in $R$.

*The operator $S$ is therefore convergent, and there exists one and only one solution of the boundary value problem if the region $R$ be taken sufficiently small in area.*

This theorem regarding the existence of a unique solution of the boundary value problem for the differential equation

$\Delta u = cu + f$ holds true also for the most general equation of elliptic type,

$$\Delta u = au_x + bu_y + cu + b.$$

The proof is however complicated by the presence of the terms in $u_x$ and $u_y$ and an investigation is necessitated regarding the behavior of the derivatives of a potential function in approaching the boundary of its region of definition.*

## § 10. THE ANALYTICAL CHARACTER OF SOLUTIONS.

It was DU BOIS REYMOND who characterized the three types of differential equations — elliptic, hyperbolic and parabolic, as "himmelweit verschieden." Their difference appears the more striking in view of later knowledge regarding the analytical character of their solutions.† To PICARD‡ is due the remarkable theorem that *every solution of a differential equation of elliptic type whose coefficients are analytic is itself analytic.* Even though the solution be required to assume non-analytic values on the boundary of a closed region it remains analytic in the interior of the region. In an important article, cited in § 4, SERGE BERNSTEIN has derived far-reaching results along these lines. He has shown that every differential equation of hyperbolic type

$$u_{xy} = F(x, y, u, u_x, u_y)$$

possesses non-analytic solutions. That the solutions of the differential equation of parabolic type

$$u_{xx} = F(x, y, u, u_x),$$

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*In PICARD's first proof of the theorem [Journal de Mathématiques, ser. 4, 6 (1890), p. 145] this investigation was omitted, and the proof was criticised on this account by DINI [Acta Mathematica, 25 (1902), p. 185] who supplied a discussion of the point in question. In a later article [Journal de Mathématiques, ser. 5, 6 (1906), p. 129] PICARD revises his former proof, restricting the nature of the bounding curve to be analytic. A more recent discussion of the behavior of the derivatives of a potential function in approaching the boundary of its region of definition is given by KELLOGG [Transactions of the American Mathematical Society, 9 (1908), p. 39].

† A function is analytic in a region if it may be expressed as a converging power series about every point of the region.

‡ Journal de l'École Polytechnique, cah. 60 (1890), §§ 1–4.

in which the first derivative of $u$ with respect to $y$ does not occur, are analytic with respect to $x$. Finally that the general non-linear equation

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

possesses only analytic solutions, provided that

$$4\frac{\partial F}{\partial u_{xx}}\frac{\partial F}{\partial u_{yy}} - \left(\frac{\partial F}{\partial u_{xy}}\right)^2 > 0.$$

The coefficients of these equations are of course supposed to be analytic.

## § 11. THE BOUNDARY VALUE PROBLEM FOR THE EQUATION $y'' + \phi(x)y = f(x)$.

The general linear equation of second order considered in § 2 may always be reduced to the form

$$(1) \qquad y'' + \phi(x)y = f(x)$$

by a change in the dependent variable. For simplicity this form will be used. In connection with the existence proof of § 2 it was remarked that the general solution of this equation has the form

$$(2) \qquad y = c_1\eta_1 + c_2\eta_2 + \eta_1\int_a^x \eta_2 f dx + \eta_2\int_x^b \eta_1 f dx$$

where $c_1$ and $c_2$ are arbitrary constants, $\eta_1$ and $\eta_2$ are linearly independent solutions of the homogeneous equation

$$(1_0) \qquad y'' + \phi(x)y = 0,$$

and are connected by the relation

$$(3) \qquad \eta_1'\eta_2 - \eta_2'\eta_1 = 1.$$

*It is required to determine a solution of the equation*

$$(1) \qquad y'' + \phi(x)y = f(x)$$

*in the interval* $(a, b)$ *which satisfies the boundary conditions*

$$(\text{I}) \qquad y(a) = A, \qquad y(b) = B.$$

On substituting the general solution (2) in equations (I) th following equations for $c_1$ and $c_2$ are obtained:

$$c_1\eta_1(a) + c_2\eta_2(a) = A - \eta_2(a)\int_a^b \eta_1 . f dx,$$

$$c_1\eta_1(b) + c_2\eta_2(b) = B - \eta_1(b)\int_a^b \eta_2 . f dx.$$

*A unique solution of the boundary value problem therefore exists if the determinant*

$$\delta = \begin{vmatrix} \eta_1(a) & \eta_2(a) \\ \eta_1(b) & \eta_2(b) \end{vmatrix}$$

*is not zero.*

*The equation* $\delta = 0$ *is the necessary and sufficient condition for the existence of a solution other than zero of the homogeneous problem*

$(1_0)$ $\quad y'' + \phi(x)y = 0, \qquad (\mathrm{I}_0) \quad y(a) = 0, \; y(b) = 0.$

Suppose $\delta = 0$ and choose $\eta_1$ as the solution of the homogeneous problem. Then a solution of the non homogeneous problem (1), (I) coexists with this solution when and only when

$$c_2\eta_2(a) = A - \eta_2(a)\int_a^b \eta_1 . f dx,$$
$$c_2\eta_2(b) = B.$$

On account of equation (3) and the equations $\eta_1(a) = 0$, $\eta_1(b) = 0$ these equations may be given the form

$$c_2 = A\eta_1'(a) - \int_a^b \eta_1 . f dx,$$
$$c_2 = B\eta_1'(b).$$

*The equation*

$$A\eta_1'(a) - B\eta_1(b) = \int_a^b \eta_1 . f dx$$

*is therefore the necessary and sufficient condition for the existence of a solution of the non-homogeneous problem* (1), (I) *when a solution* $\eta_1$ *of the homogeneous problem* $(1_0)$, $(\mathrm{I}_0)$ *exists, i. e., when* $\delta = 0$.

*If this condition is satisfied the solution of* (1), (I) *is not uniquely determined, but has the form*

$$y = \bar{y} + c\eta_1,$$

*where $\bar{y}$ is one such solution and $c$ is an arbitrary constant.*

If $A = 0$, $B = 0$ the above condition becomes

$$\int_a^b \eta_1 f dx = 0.$$

This case is of special importance for the developments of § 13.

Similar theorems hold for the general equation

$$y'' + p(x)y' + q(x)y = f(x),$$

and the boundary conditions

$$a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = A,$$
$$b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = B,$$

provided that certain relations are satisfied by the coefficients $a_i$, $b_i$.*

## § 12. THE TRANSVERSE VIBRATIONS OF A CORD.

The small transverse vibrations of a stretched cord are to be determined as the solution $u$ of the differential equation

$$(1) \qquad A(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

where $t$ is the time, $x$ the distance along the cord in its position of equilibrium, and $A(x)$ a function depending on the nature of the cord and on the tension. Suppose the cord is fixed at its ends, $x = a$, $x = b$, is initially distorted so that $u(x, 0) = f(x)$ is given, and is then allowed to vibrate freely from rest. Then the function $u$ must satisfy the further conditions

$$(2) \quad u(x, 0) = f(x), \qquad (3) \quad u_t(x, 0) = 0,$$
$$(4) \quad u(a, t) = 0, \qquad (5) \quad u(b, t) = 0.$$

In accordance with the usual method, assume that $u$ is the product of a function of $x$ alone and a function of $t$ alone,

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*See an article by the writer in the Transactions of the American Mathematical Society, 7 (1906), p. 337.

$$u(x,\, t) = y(x)T(t).$$

Then the differential equation becomes

$$Ay\frac{d^2T}{dt^2} = T\frac{d^2y}{dx^2}.$$

After separating the variables by division each member of the equation must be a constant, say $-\lambda$, and the equations

$$T'' + \lambda T = 0, \qquad y'' + \lambda A(x)y = 0$$

result. On account of (3 the solution

$$T = \cos t\sqrt{\lambda}$$

is to be chosen for the first equation. The second equation is to be solved under the conditions

$$y(a) = 0, \qquad y(b) = 0.$$

A first problem is then to determine the constant $\lambda$ so that such a solution exists. Suppose $\lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots$ form a set of such values, and $y_1,\ y_2,\ y_3,\ \cdots$ are the corresponding solutions. Then the function

$$u = \Sigma c_i y_i(x) \cos t\sqrt{\lambda_i}$$

is a solution of (1), (3), (4), (5), where $c_i$ are arbitrary constants. The original problem will then be solved, if the constants $c_i$ may be so determined that

$$\Sigma c_i y_i(x) = f(x).$$

## § 13. THE EXISTENCE OF NORMAL FUNCTIONS FOR THE EQUATION $y'' + \lambda A(x)y = 0$.*

We shall consider the problem indicated in the preceding section. It is required to determine the parameter $\lambda$ in such a way that a solution, not identically zero, of the equation

$$y'' + \lambda A(x)y = 0 \tag{1}$$

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* The developments of this section are included in the article cited at the end of § 11. This differential equation has formed the subject of many investigations, the more important of which are cited in the article mentioned above.

will exist, which satisfies the conditions

$$(2) \qquad y(a) = 0, \quad y(b) = 0.$$

It was seen in § 11 that a solution of this problem exists provided that the determinant

$$\delta = \begin{vmatrix} \eta_1(a) & \eta_2(a) \\ \eta_1(b) & \eta_2(b) \end{vmatrix}$$

is zero. In this case the solutions $\eta_1$, $\eta_2$ of (1) are functions of $\lambda$, expressed as power series in $\lambda$ which converge for all values of $\lambda$. This is seen at once from the form of solution obtained in § 2. The same is then true of $\delta = \delta(\lambda)$. A solution of the required type is possible only if $\lambda$ is a zero of this function. These zeros will be called *normal parameter values*, and the corresponding solutions of equations (1), (2) *normal functions*.

It will be shown that to a known set of independent normal functions an additional normal function, linearly independent of those in the known set, may always be found. It is to be noticed that the proof requires no modification for the case where the known set consists of no functions at all. With a complete induction proof it then combines the proof of the existence of the first function, and the existence of an infinite series of normal functions is thus assured.

Suppose the functions

$$y_1, \; y_2, \; \cdots, \; y_{n-1}$$

are normal functions for the positive parameter values

$$\lambda_1 < \lambda_2 < \lambda_3 \cdots < \lambda_{n-1}.$$

Let the arbitrary constant factor of each function $y_i$ be so determined that

$$(3) \qquad \int_a^b A y_i^2 dx = 1.$$

This is possible, since from the equations (2) and

$$(4) \qquad y_i'' + \lambda_i A y_i = 0$$

it follows that

$$\lambda_i \int_a^b A y_i^2 dx = -\int_a^b y_i y_i'' dx = \int_a^b y_i'^2 dx > 0,$$

and by assumption, $\lambda_i > 0$. Furthermore, from equation (4) and the similar equation for the index $k$, the equations

$$(\lambda_i - \lambda_k)\int_a^b A y_i y_k dx = \int_a^b (y_i'' y_k - y_k'' y_i) dx$$

$$= \int_a^b (y_k' y_i' - y_i' y_k') dx = 0$$

are deduced, and therefore the equation

$$\int_a^b A y_i y_k dx = 0 \tag{5}$$

holds for unequal values of $i$ and $k$.

The existence of a new normal function $y_n$ will now be proved in the following manner:[*] *Let $\lambda_n$ be defined as the lower limit of the values of the integral*

$$J = \int_a^b y'^2 dx$$

*formed for functions $y$ which satisfy the following conditions:*

$$(\alpha)\quad \begin{cases} (\alpha_1) & y(a) = 0, \quad y(b) = 0, \\ (\alpha_2) & \int_a^b A y^2 dx = 1, \\ (\alpha_3) & \int_a^b A y_i y dx = 0 \qquad (i = 1, 2, \ldots, n-1). \end{cases}$$

Since the values of $J$ are all positive such a lower limit must exist, even though it is not *a priori* evident that a function exists which satisfies condition ($\alpha$) and gives to the integral $J$ the value $\lambda_n$. It will be shown that for $\lambda = \lambda_n$ a solution of the boundary problem exists which is independent of the known solution $y_i$, i. e., that $\lambda_n$ is a normal parameter value, a root of $\delta(\lambda) = 0$. It will also appear that this new normal function $y_n$ gives the value $\lambda_n$ to the integral $J$ and satisfies conditions ($\alpha$), so that it is the solution of a minimum problem. The proof of the former statement will consist in showing that on assumption of the contrary a function may be found which satisfies conditions ($\alpha$) and gives to $J$ a value which is less than $\lambda_n$, its least possible value consistent with these conditions.

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[*] Based upon an article by H. WEBER [Mathematische Annalen, 1 (1869), p. 1], on the partial differential equation $\Delta u + \lambda A u = 0$, where reasoning similar to that of DIRICHLET's principle is employed.

Let

$$u_1, u_2, u_3, \cdots$$

be an infinite series of approximating functions for the problem of giving $J$ its least value consistent with conditions ($\alpha$), i. e., the $u_i$ form a set of functions which satisfy conditions ($\alpha$), and for which*

$$\lim_{i=\infty} J(u_i) = \lambda_n. \tag{8}$$

Define a set of functions $f_i$ by the equations

$$u_i'' + \lambda_n A u_i = f_i. \tag{9}$$

Multiply these equations by $u_i$ and integrate from $a$ to $b$. On account of conditions ($\alpha$) the result is

$$\lambda_n - J(u_i) = \int_a^b f_i u_i dx.$$

It therefore follows from (8), that

$$\lim_{i=\infty} \int_a^b f_i u_i dx = 0. \tag{10}$$

Suppose now that $\lambda_n$ is not a normal value distinct from $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$. Then either it is not any normal value, or else it coincides with one of the known set. In the first case a solution of the equation

$$y'' + \lambda A y = f$$

under the boundary condition ($\alpha_1$) exists, whatever be $f$; in the second case such a solution exists, according to § 2, if $f$ satisfies the condition

$$\int_a^b A f y_k dx = 0,$$

where $k$ is the index of the normal value with which $\lambda_n$ coincides. Now on account of condition $(\alpha_3)$ the equations

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* It may be assumed without restriction that the functions $u_i$ have continuous second derivatives.

$$(11) \qquad \int_a^b Ay_iu_h dx = 0 \qquad (i = 1, 2, \cdots, n-1)$$

hold for all values of $h$. There therefore exists a solution of the equation

$$(12) \qquad v_h'' + \lambda_n A v_h = Au_h$$

under conditions $(\alpha_1)$ for each value of $h$. It is these solutions which, when combined with the approximating functions $u_h$ of the minimum problem, show the impossibility of the assumption that $\lambda_n$ is not a new normal parameter value.

From (12) and the equations

$$y_i'' + \lambda_i A y_i = 0$$

it follows, on account of (11) and the boundary conditions, that

$$(\lambda_n - \lambda_i) \int_a^b Av_hy_i dx = \int_a^b (y_i''v_h - v_h''y_i) dx = 0,$$

and therefore each function $v_h$ satisfies the condition $(\alpha_3)$, unless $\lambda_n = \lambda_i$. But if $\lambda_n = \lambda_i$ the functions $v_h$ are not uniquely determined by equations (12) and $(\alpha_1)$ but have the form (see § 2),

$$v_h = V_h + c_h y_i,$$

where $c_h$ are arbitrary constants, and $V_h$ some particular solutions of (12) and $(\alpha_1)$. The value of $c_h$ may be so determined that

$$\int_a^b Av_hy_i dx = \int_a^b A V_h y_i dx + c_h = 0.$$

The functions $v_h$ will then satisfy condition $(\alpha_3)$ in all cases.

Consider the functions

$$w_h = u_h + cv_h$$

where $c$ is a constant. These functions satisfy $(\alpha_1)$ and $(\alpha_3)$. On multiplying the equation

$$w_h'' + \lambda_n A w_h = f_h + cAu_h,$$

a consequence of the equations (9) and (12), by $w_h = u_h + cv_h$

and integrating from $a$ to $b$, the equation

$$\lambda_n \int_a^b A w_h^2 dx - J(w_h) = \int_a^b f_h u_h dx + c \int_a^b (f_h v_h + A u_h^2) dx$$
$$+ c^2 \int_a^b A u_h v_h dx$$

is obtained. From equations (9), (11) and the boundary conditions it follows that

$$\int_a^b (f_h u_h - A u_h^2) dx = \int_a^b (v_h'' u_h - u_h'' v_h) dx = 0,$$

and therefore that

$$\int_a^b f_h u_h dx = \int_a^b A u_h^2 dx = 1.$$

Hence

$$\lambda_n \int_a^b A w_h^2 dx - J(w_h) = \int_a^b f_h u_h dx + 2c + c^2 \int_a^b A u_h v_h dx.$$

Consider the integral in the last term of the second member. It will be shown that the absolute value of these integrals for all values of $h$ are less than a fixed number $B$, independent of $h$. This is manifestly true if it holds for the maxima of $|u_h|$ and $|v_h|$. The expression for $v_h$ as solutions of equations (12) obtained by the method of variation of parameters shows that the maxima of $|v_h|$ remain under a fixed limit if this is true of the maxima of $|u_h|$. Now

$$|u_h| = \left| \int_a^x u_h' dx \right| < \int_a^b |u_h'| dx < \int_a^b (u_h'^2 + 1) dx = J(u_h) + (b-a).$$

Since the values of $J(u_h)$ approach the limit $\lambda_n$ it follows that there exists a fixed number $B$ such that

$$\left| \int_a^b A u_h v_h dx \right| < B$$

for all values of $h$. Therefore

$$\lambda_n \int_a^b A w_h^2 dx - J(w_h) \geqq \int_a^b f_h u_h dx + 2c - c^2 B.$$

$2c - c^2B = \delta^2$. Then on account of (10) $h$ may be taken so large that

$$\left|\int_a^b f_h u_h dx\right| < \delta^2,$$

and therefore

$$J(w_h) < \lambda_n \int_a^b A w_h^{\,2} dx. \tag{13}$$

We may therefore write

$$\int_a^b A w_h^{\,2} dx = m^2$$

where $m$ is a real number, for $J(w_h)$ is positive and $\lambda_n$ is not negative. Put

$$y = \frac{w_h}{m}.$$

Then

$$\int_a^b A y^2 dx = \frac{1}{m^2}\int_a^b A w_h^{\,2} dx = 1,$$

and therefore $y$ satisfies all conditions $(\alpha)$. Furthermore from equation (13)

$$J(y) < \lambda_n.$$

But this is impossible, since $\lambda_n$ is the lower limit of the values of $J$ under conditions $(\alpha)$. It follows that the assumption, under which the functions $v_h$ and $w_h$ were shown to exist, is false, and $\lambda_n$ is therefore a normal parameter value, distinct from the known values $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$. The corresponding normal function $y_n$ may be multiplied by a constant so that the equation

$$\int_a^b A y_n^{\,2} dx = 1$$

holds—as was seen before in the case of the known function $y_i$. Furthermore from the equations

$$y_i'' + \lambda_i A y_i = 0,$$
$$y_n'' + \lambda_n A y_n = 0,$$

it follows, since $\lambda_n \neq \lambda_i$ that the function $y_n$ satisfies conditions $(\alpha_3)$. On multiplying the second of the above equations by $y_n$

$$\lambda_n = -\int_a^b y_n'' y_n dx = \int_a^b y_n'^2 dx = J(y_n).$$

*The normal function $y_n$ satisfies conditions $(\alpha)$ and gives the value $\lambda_n$ to the integral $J$.*

The existence of an infinite set of normal functions $y_i$ corresponding to positive normal parameter values $\lambda_i$ is thus proved. If $A$ changes sign in the interval $(a, b)$ an exactly similar consideration, whereby the condition

$$\int_a^b Ay^2 dx = 1$$

is replaced by

$$\int_a^b Ay^2 dx = -1,$$

shows the existence of an infinite series of normal functions corresponding to negative normal parameter values. Since the normal parameter values $\lambda_i$ are the roots of an integral transcendental function $\delta(\lambda) = 0$ they can have no finite limiting point. The following theorem may therefore be stated:

*There exists an infinite series of normal parameter values $\lambda_n$ and corresponding solutions (normal functions) $y_n$ of the differential equation*

$$(1) \qquad y'' + \lambda A y = 0,$$

*and the boundary conditions*

$$(2) \qquad y(a) = 0, \qquad y(b) = 0.$$

*If $A$ changes sign in $(a, b)$ the values $\lambda_n$ include an infinite series of positive terms $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, increasing without limit, and an infinite series of negative terms $\lambda_{-1} > \lambda_{-2} > \lambda_{-3} > \cdots$, decreasing without limit. The function $y_n$ satisfies the conditions* *

$$\int_a^b PAy^2 dx = \pm 1, \qquad \int_a^b PAy_i y dx = 0$$

$$[i = \pm 1, \pm 2, \cdots, \pm(n-1)],$$

*and gives to the integral*

$$J(y) = \int_a^b y'^2 dx$$

---

* The upper or lower signs are to be taken according as $n$ is positive or negative.

*its least possible value consistent with these conditions and with equations* (2). *This minimum value of $J$ is* $\pm \lambda_n$.

## § 14. THE EXPANSION OF A FUNCTION IN TERMS OF NORMAL FUNCTIONS.*

On the basis of the theorem stated at the end of the preceding section the following theorem will be proved:

*A function $f(x)$ which vanishes at $a$ and $b$, is continuous in $(a, b)$ and has a derivative which is continuous except at a finite number of points, may be expanded in a uniformly and absolutely converging series of the form*

$$\sum_{i=-\infty}^{\infty} c_i y_i,$$

*where*

$$c_{\pm i} = \pm \int_a^b f A y_{\pm i} dx.$$

A lemma regarding the constants $c_i$ will first be proved, namely, that the series of numbers

$$\sum_{i=1}^{\infty} c_i^2 \lambda_i, \qquad \sum_{i=-1}^{-\infty} c_i^2 \lambda_i,$$

are convergent.† The following equation is a result of the rties of the functions $y_i$:

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* Recent articles of special importance on this subject are those of HILBERT, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, 1904, 1905, 1906, and KNESER, Mathematische Annalen, vol. 63 (1907), p. 477. These articles are based on the theory of integral equations, as developed by HILBERT and SCHMIDT. KNESER removes the restrictions imposed by HILBERT, that the function be continuous with its first and second derivatives, and satisfy the same boundary conditions as the normal functions. In all work on this subject the boundary conditions for the normal functions are special cases of equations (2), above, except in the important case of singularities of the differential equation. The function $A$ is supposed positive in all previous work, except in the fifth memoir of HILBERT's series, in which, by considering a new class of integral equations, the restriction is removed. This restriction will not be made in the following treatment, which is a special case of the discussion given by the writer in the Transactions of the American Mathematical Society, vol. 8 (1907), p. 427. A very general expansion problem is treated by BIRKHOFF, ibid., vol. 9, (1908), p. 373.

† Cf. SCHMIDT, Mathematische Annalen, vol. 63 (1907), pp. 439, 440.

$$\int_a^b \left(f - \sum_{i=1}^n c_i y_i\right)'^2 dx = \int_a^b f'^2 dx + \sum_{i=1}^n c_i^2 \lambda_i$$

$$- 2c_i \sum_{i=1}^n \int_a^b f' y_i' dx + \sum_{\substack{i,\, k=1 \\ (i \neq k)}}^n \int_a^b y_i' y_k' dx.$$

the differential equation for $y_i$ and the boundary condi-
it follows that

$$\int_a^b f' y_i' dx = - \int_a^b f y_i'' dx = \lambda_i \int_a^b f A y_i dx = \lambda_i c_i,$$

$$\int_a^b y_k' y_i' dx = - \int_a^b y_k y_i'' dx = \lambda_i \int_a^b A y_k y_i dx = 0.$$

the first member of equation (3) is positive, it follows
this and the above equations, that

$$\int_a^b f'^2 dx \geqq \sum_{i=1}^n c_i^2 \lambda_i.$$

holds for all values of $n$, and since the terms of the series
ng the second member are all positive, this series is con-
nt. The convergence of the series with negative sub-
s is proved in the same way.

e convergence of the series of functions may now be easily
1. Consider the terms with positive subscripts, and write

$$\phi_{n,m} = \sum_{i=n}^m |c_i y_i| = \sum_{i=n}^m \left| c_i \int_a^x y_i' dx \right|.$$

$$(\phi_{n,m})^2 \leqq 2 \sum_{i=n}^m c_i^2 \left(\int_a^x y_i' dx\right)^2,$$

$\alpha^2 + \beta^2 \geqq 2\alpha\beta$ for any numbers $\alpha$, $\beta$. Furthermore, since
alue of

$$\int_a^x (\mu + y_i')^2 dx = (x - a)\mu^2 + 2\mu \int_a^x y_i' dx + \int_a^x y_i'^2 dx$$

itive for all values of $\mu$, the discriminant is not negative, and therefore

$$\left(\int_a^b y_i'dx\right)^2 \leqq (x-a)\int_a^x y_i'^2dx \leqq (b-a)\lambda_i.$$

Hence

$$(\phi_{n,m})^2 \leqq (b-a)\sum_{i=n}^{m} c_i^2\lambda_i,$$

and it follows from the preceding lemma that the series

$$c_1y_1 + c_2y_2 + \cdots$$

converges uniformly and absolutely. The convergence of the series of terms with negative subscripts is proved in the same way.

It will now be shown, by the aid of the minimum properties of the normal functions, that the series represents the function. Since the series converges uniformly, the difference

$$f - \sum_{i=-\infty}^{i=\infty} c_iy_i = g$$

is a continuous function. On multiplying this equation by $Ay_i$ and integrating from $a$ to $b$ the equation

$$(2) \qquad \int_a^b Ay_igdx = 0$$

results, whatever be $i$. This equation is sufficient to insure that $g$ is identically zero. It will be shown that on assumption of the contrary a function $y$ may be formed which has a continuous derivative and satisfies the boundary conditions and the equations*

$$(3) \qquad \int_a^b Ay_iydx = 0 \quad (i = \pm 1, \pm 2, \pm 3, \cdots),$$

$$(4) \qquad \int_a^b Ay^2dx = \pm 1.$$

Suppose such a function exists. The value of the integral

$$J = \int_a^b y'^2dx$$

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* I. e., $y$ will satisfy (3), (4) either with the upper signs throughout, or else with the lower signs.

is certainly less than $\lambda_n$ and $-\lambda_{-n}$ if $n$ be taken sufficiently large, since these values increase beyond limit with $n$. Now $y$ satisfies all the conditions of the minimum problem of which either $y_n$ or else $y_{-n}$ is the solution, according as (4) holds with upper or lower sign. But it gives to the integral $J$ a value less than the smallest value $\lambda_n$ or $-\lambda_{-n}$ consistent with the conditions. Such a function can therefore not exist.

It remains to show that if the difference $g$ is not identically zero a function $y$ satisfying the conditions $y(a) = 0$, $y(b) = 0$ and equations (3) and (4) can be found. Consider the function $u$ satisfying the equation

$$u'' = Ay$$

and vanishing at $a$ and at $b$. This function has a continuous second derivative. Now

$$y_i'' + \lambda_i A y_i = 0,$$

and therefore, on account of (2),

$$\int_a^b (u''y_i - y_i''u)dx = -\lambda_i \int_a^b Ay_i u dx.$$

The left member vanishes after an integration by parts, since $y_i$ and $u$ vanish at $a$ and $b$, and therefore $u$ satisfies (3). On multiplying $u$ by a constant equation (4) may be satisfied also, unless

$$\int_a^b Au^2 dx = 0.$$

Suppose however that this equation holds. Solve the equation

$$v'' = Au$$

under the boundary conditions. This function satisfies (3), the proof being the same as before except that $g$ is replaced by $u$. Then the desired function $y$ may be formed by multiplying $v$ by a constant, unless

$$\int_a^b Av^2 dx = 0.$$

Suppose again that this is the case. Consider then the function

$$w = u - v,$$

which satisfies (3) and vanishes at $a$ and $b$. Then

$$\int_a^b Aw^2dx = -2\int_a^b Auvdx = -2\int_a^b vv''dx = 2\int_a^b v'^2dx$$

The required function $y$ may therefore be formed in any unless $g$ is identically zero. It follows that the series sents the function.